M1 General Physics

Particles

Cross-sections

1 Invariant one-particle phase-space

1. Show that

$$\frac{d^3\vec{p}}{(2\pi)^3 2E(|\vec{p}|)} = \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2 - m^2)\theta(p_0)$$
(1)

where $E = E(|\vec{p}|) = \sqrt{\vec{p}^2 + m^2}$, and conclude about the Lorentz invariance of this oneparticle phase-space. *Hint*: use the fact that

$$\delta(f(x)) = \sum_{i} \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{2}$$

where x_i are the simple roots of f(x).

2. Write $d^3\vec{p}$ in terms of $p = |\vec{p}|$ (beware to this rather standard notation: p here should not be confused with the 4-momentum!) and of the elementary solid angle $d^2\Omega$.

3. Write $d^2\Omega$ in spherical coordinates.

2 Phase space in the center-of-mass frame

We consider the $2 \to 2$ process $A(p_A) B(p_B) \to C(p_C) D(p_D)$, where A, B, C, D are particles of mass respectively equal to m_A, m_B, m_C, m_D . Our aim is to simplify the expression of the phase space

$$d(P.S) = (2\pi)^4 \delta^{(4)} (p_A + p_B - p_C - p_D) \frac{d^3 p_C}{(2\pi)^2 2E_C} \frac{d^3 p_D}{(2\pi)^2 2E_D}$$
(3)

in the center-of-mass frame. We denote $p_C = |\vec{p}_C|$ and $p_D = |\vec{p}_D|$. One may use the Mandelstam variable $s = (p_A + p_B)^2$. In the center-of-mass frame, we denote $p_f^* = p_C$.

1. Show that in the center-of-mass frame,

$$d(P.S) = \frac{1}{4\pi^2} \delta^{(3)}(\vec{p}_C + \vec{p}_D) \,\delta(E_C(p_C) + E_D(p_D) - \sqrt{s}) \,\frac{d^3 p_C}{2E_C(p_C)} \frac{d^3 p_D}{2E_D(p_D)},\tag{4}$$

and give the expressions of $E_C(p_C)$ and $E_D(p_D)$.

2. Show finally that

$$d(P.S) = \frac{1}{4\pi^2} \frac{p_f^*}{4\sqrt{s}} d^2 \Omega \,.$$
(5)

Hint: one may use Eq. (2).

3 Study of the "spinless" electron-muon scattering

Consider "spinless" electron-muon scattering. Denote θ the scattering angle in the center-ofmass system (c.m.s), i.e. the angle between the outgoing and incoming electron (or muon) momentum. One may use the notation of section 1, with $m_e = m_A = m_C$ and $m_\mu = m_B = m_D$.

1. Write the expression of the scattering amplitude \mathcal{M} .

2. We denote $s = (p_A + p_B)^2$. In the c.m.s., write the equations satisfied by \vec{p}_A , \vec{p}_B , \vec{p}_C , \vec{p}_D and E_A , E_B , E_C , E_D . Deduce an equation satisfied by $|\vec{p}_A|$ and $|\vec{p}_C|$ and conclude about their relative magnitude.

Then, write the energy and space components of p_A , p_B , p_C , p_D in terms of $s = (p_A + p_B)^2$ and of E_A , E_B , \vec{p}_A and \vec{p}_C .

3. Give the expression of q^2 as a function of θ and $|\vec{p}_A|$. Then, write q^2 in terms of $s = (p_A + p_B)^2$, m_A , m_B . One may use the obtained expression for $|\vec{p}_A|$ in the 2020 mid-term exam, or directly solve the equation satisfied by $|\vec{p}_A|$ in question 2.

4. Express the numerator of \mathcal{M} as a function of E_A , E_B , \vec{p}_A^2 and $\cos\theta$. Write E_A and E_B in terms of s, m_A , m_B (one may rely on results obtained in the 2020 mid-term exam) and show finally that

$$\mathcal{M} = e^2 \left[\frac{3 + \cos\theta}{1 - \cos\theta} + \frac{C}{1 - \cos\theta} \right] \tag{6}$$

where C is a function of s, m_A, m_B which vanishes in the high-energy limit.

5. Prove finally that the differential cross-section reads

$$\left. \frac{d\sigma}{d\Omega} \right|_{cms} = \frac{\alpha^2}{4s} \left(\frac{3 + C + \cos\theta}{1 - \cos\theta} \right)^2,\tag{7}$$

where $\alpha = e^2/(4\pi)$ is the fine-structure constant.