## M1 General Physics

## Particles

## Cross-sections

## 1 Invariant one-particle phase-space

1. Show that

$$
\begin{equation*}
\frac{d^{3} \vec{p}}{(2 \pi)^{3} 2 E(|\vec{p}|)}=\frac{d^{4} p}{(2 \pi)^{4}}(2 \pi) \delta\left(p^{2}-m^{2}\right) \theta\left(p_{0}\right) \tag{1}
\end{equation*}
$$

where $E=E(|\vec{p}|)=\sqrt{\vec{p}^{2}+m^{2}}$, and conclude about the Lorentz invariance of this oneparticle phase-space.
Hint: use the fact that

$$
\begin{equation*}
\delta(f(x))=\sum_{i} \frac{\delta\left(x-x_{i}\right)}{\left|f^{\prime}\left(x_{i}\right)\right|} \tag{2}
\end{equation*}
$$

where $x_{i}$ are the simple roots of $f(x)$.
2. Write $d^{3} \vec{p}$ in terms of $p=|\vec{p}|$ (beware to this rather standard notation: $p$ here should not be confused with the 4-momentum!) and of the elementary solid angle $d^{2} \Omega$.
3. Write $d^{2} \Omega$ in spherical coordinates.

## 2 Phase space in the center-of-mass frame

We consider the $2 \rightarrow 2$ process $A\left(p_{A}\right) B\left(p_{B}\right) \rightarrow C\left(p_{C}\right) D\left(p_{D}\right)$, where $A, B, C, D$ are particles of mass respectively equal to $m_{A}, m_{B}, m_{C}, m_{D}$. Our aim is to simplify the expression of the phase space

$$
\begin{equation*}
d(P . S)=(2 \pi)^{4} \delta^{(4)}\left(p_{A}+p_{B}-p_{C}-p_{D}\right) \frac{d^{3} p_{C}}{(2 \pi)^{2} 2 E_{C}} \frac{d^{3} p_{D}}{(2 \pi)^{2} 2 E_{D}} \tag{3}
\end{equation*}
$$

in the center-of-mass frame. We denote $p_{C}=\left|\vec{p}_{C}\right|$ and $p_{D}=\left|\vec{p}_{D}\right|$. One may use the Mandelstam variable $s=\left(p_{A}+p_{B}\right)^{2}$. In the center-of-mass frame, we denote $p_{f}^{*}=p_{C}$.

1. Show that in the center-of-mass frame,

$$
\begin{equation*}
d(P . S)=\frac{1}{4 \pi^{2}} \delta^{(3)}\left(\vec{p}_{C}+\vec{p}_{D}\right) \delta\left(E_{C}\left(p_{C}\right)+E_{D}\left(p_{D}\right)-\sqrt{s}\right) \frac{d^{3} p_{C}}{2 E_{C}\left(p_{C}\right)} \frac{d^{3} p_{D}}{2 E_{D}\left(p_{D}\right)}, \tag{4}
\end{equation*}
$$

and give the expressions of $E_{C}\left(p_{C}\right)$ and $E_{D}\left(p_{D}\right)$.
2. Show finally that

$$
\begin{equation*}
d(P . S)=\frac{1}{4 \pi^{2}} \frac{p_{f}^{*}}{4 \sqrt{s}} d^{2} \Omega . \tag{5}
\end{equation*}
$$

Hint: one may use Eq. (2).

## 3 Study of the "spinless" electron-muon scattering

Consider "spinless" electron-muon scattering. Denote $\theta$ the scattering angle in the center-ofmass system (c.m.s), i.e. the angle between the outgoing and incoming electron (or muon) momentum. One may use the notation of section 1 , with $m_{e}=m_{A}=m_{C}$ and $m_{\mu}=m_{B}=m_{D}$.

1. Write the expression of the scattering amplitude $\mathcal{M}$.
2. We denote $s=\left(p_{A}+p_{B}\right)^{2}$. In the c.m.s., write the equations satisfied by $\vec{p}_{A}, \vec{p}_{B}, \vec{p}_{C}, \vec{p}_{D}$ and $E_{A}, E_{B}, E_{C}, E_{D}$. Deduce an equation satisfied by $\left|\vec{p}_{A}\right|$ and $\left|\vec{p}_{C}\right|$ and conclude about their relative magnitude.
Then, write the energy and space components of $p_{A}, p_{B}, p_{C}, p_{D}$ in terms of $s=\left(p_{A}+p_{B}\right)^{2}$ and of $E_{A}, E_{B}, \vec{p}_{A}$ and $\vec{p}_{C}$.
3. Give the expression of $q^{2}$ as a function of $\theta$ and $\left|\vec{p}_{A}\right|$. Then, write $q^{2}$ in terms of $s=$ $\left(p_{A}+p_{B}\right)^{2}, m_{A}, m_{B}$. One may use the obtained expression for $\left|\vec{p}_{A}\right|$ in the 2020 mid-term exam, or directly solve the equation satisfied by $\left|\vec{p}_{A}\right|$ in question 2 .
4. Express the numerator of $\mathcal{M}$ as a function of $E_{A}, E_{B}, \vec{p}_{A}^{2}$ and $\cos \theta$. Write $E_{A}$ and $E_{B}$ in terms of $s, m_{A}, m_{B}$ (one may rely on results obtained in the 2020 mid-term exam) and show finally that

$$
\begin{equation*}
\mathcal{M}=e^{2}\left[\frac{3+\cos \theta}{1-\cos \theta}+\frac{C}{1-\cos \theta}\right] \tag{6}
\end{equation*}
$$

where $C$ is a function of $s, m_{A}, m_{B}$ which vanishes in the high-energy limit.
5. Prove finally that the differential cross-section reads

$$
\begin{equation*}
\left.\frac{d \sigma}{d \Omega}\right|_{c m s}=\frac{\alpha^{2}}{4 s}\left(\frac{3+C+\cos \theta}{1-\cos \theta}\right)^{2} \tag{7}
\end{equation*}
$$

where $\alpha=e^{2} /(4 \pi)$ is the fine-structure constant.

