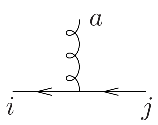
M2/CFP/Parcours of Physique Théorique
Invariances in physics and group theory

Graphical rules for $SU(N)$

In this problem, we deal with a graphical language, which turns out to be very useful when computing group factors in a Yang-Mills field theory constructed on the gauge group $SU(N)$. This applies in particular to QCD, for which, although $N = 3$, it turns out to be useful to consider the extension to $SU(N)$ for reasons which will become clear in section 3.2.

1 Generators

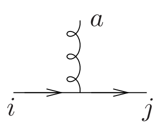
A graphical representation of the Lie algebra generators in the fundamental representation t^a_{ij} is given by



$$\equiv \quad t^a_{ij}\,. \tag{1}$$

The fundamental lines carry arrows to distinguish the two representations N and $\bar{N}$ which are not equivalent for $N \geq 3$.

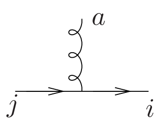
The graphical representation for the generator $(t^a)^*_{ij}$ is given by



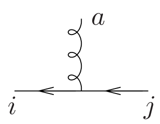
$$\equiv \quad (t^a)^*_{ij}\,. \tag{2}$$

Since t^a is hermitian,

$$(t^a)^\dagger_{ij} \quad =$$

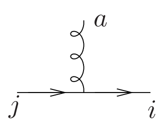


$$= \quad t^a_{ij} \quad =$$



(3)

One can of course, through a turn around by π, write



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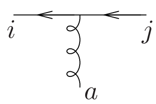
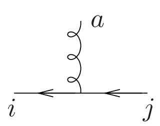


(4)

thus hermiticity (3) implies that



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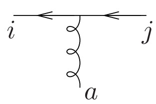


(5)

After setting these rules, a given ordered product of generators drawn graphically can be translated algebraically *following the arrows backward*, as one would do for Dirac γ matrices.

The generators are normalized conventionally through the relation

$$\boxed{\mathrm{Tr}\,(t^a\,t^b) = \tfrac{1}{2}\delta^{ab}\,.} \tag{6}$$

1. Derive the Fierz identity

$$\boxed{t^a_{ij}\,t^a_{k\ell} = \frac{1}{2}\left(\delta_{i\ell}\,\delta_{jk} - \frac{1}{N}\delta_{ij}\,\delta_{k\ell}\right)\,.} \tag{7}$$

[**Solution** : The basis of the proof is identical to the one used for spinors: I ans t^a $(a = 1, \cdots, N^2-1)$ provide a basis for hermitian matrices $N \times N$. For a given arbitrary hermitian matrix A, one can thus write

$$A = c^0\,I + c^a\,t^a\,. \tag{8}$$

Using (6), one gets the expression of the coefficients appearing in the decompostion (8), in the form

$$c^0 = \frac{\mathrm{Tr}A}{N} \quad \text{et} \quad c^a = 2\,\mathrm{Tr}[t^a\,A]\,. \tag{9}$$

This leads to write (8) in the form, focusing to the coefficient ij,

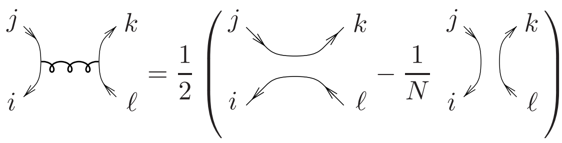
$$\frac{\mathrm{Tr}A}{N}\delta_{ij} + 2\,t^a_{k\ell}\,A_{\ell k}\,t^a_{ij} = A_{ij}$$

i.e., for every $A_{\ell k}$,

$$A_{\ell k}\left[\frac{1}{N}\delta_{k\ell}\,\delta_{ij} + 2\,t^a_{ij}\,t^a_{k\ell} - \delta_{i\ell}\,\delta_{jk}\right] = 0$$

which ends the proof after identifying the coefficients of $A_{\ell k}$.]

2. Show that this identity reads graphically:

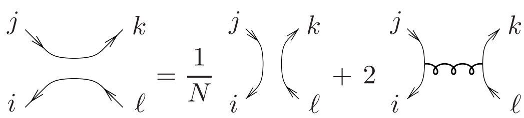


(10)

[**Solution** : This result is obvious from the previous question.]

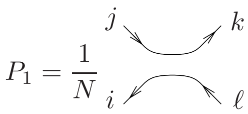
3. From the decomposition of identity acting in the tensor product space $N \otimes \bar{N}$, Find the graphical rules for the singlet and adjoint projectors.

[**Solution** : The decomposition of identity acting in the tensor product space $N \otimes \bar{N}$ is

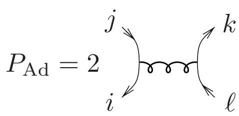


(11)

We thus have



(12)



(13)

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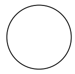
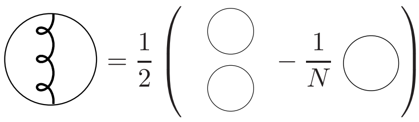
4. Check that these projectors give the appropriate values for the dimension of the singlet and adjoint representations.

[**Solution** :

$$\dim_1 = \mathrm{Tr}P_1 = \frac{1}{N}$$  $$= 1. \qquad (14)$$

$$\dim_{Ad} = \mathrm{Tr}P_{\mathrm{Ad}} = 2$$ 

$$. \qquad (15)$$

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2 A few applications

2.1 Some color factors in fundamental representation

Let us consider now a few typical color factors, which we will encounter several times.

1. Show that $(t^a\, t^a)_{ij} = C_F\, \delta_{ij}$ and compute C_F .

[**Solution** : This result is expected: $t^a t^a$ is a Casimir for $SU(N)$, thus from Schur lemma it should be a multiple of identity. In order to get this multiplicative factor C_F, it is enough to compute the trace of $t^a t^a$, which is equal to $C_F\, N$ ($\mathrm{Tr}\, I = N$). Since $\mathrm{Tr}(t^a t^a) = \frac{1}{2}\delta^{aa} = \frac{N^2-1}{2}$ (number of generators= $N^2 - 1$), thus $C_F = \frac{N^2-1}{2N}$.
]
2. Derive the same result using the Fierz identity

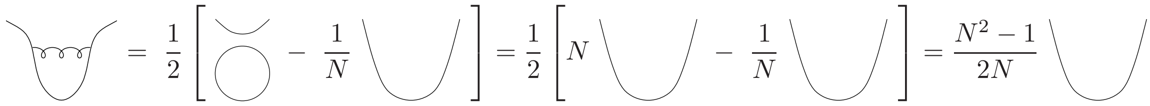
a. Algebraically

[**Solution** :

$$(t^a\, t^a)_{ij} = t^a_{ik}\, t^a_{k\ell} = \sum_k \frac{1}{2}\left(\delta_{ij}\delta_{kk} - \frac{1}{N}\delta_{ik}\,\delta_{kj}\right) = \frac{1}{2}\left(N - \frac{1}{N}\right)\delta_{ij} = \frac{N^2-1}{2N}\delta_{ij}\,.$$

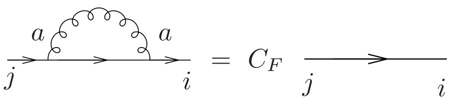
]
b. Graphically.

[**Solution** :



Thus

$$\boxed{(t^a\, t^a)_{ij} = \frac{N^2-1}{2N}\,\delta_{ij} \equiv C_F\,\delta_{ij} \qquad \text{i.e.}} \qquad (16)$$



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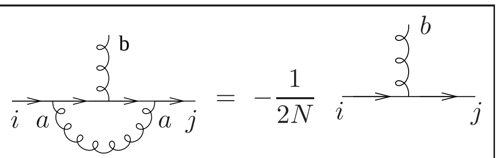
3. Derive that

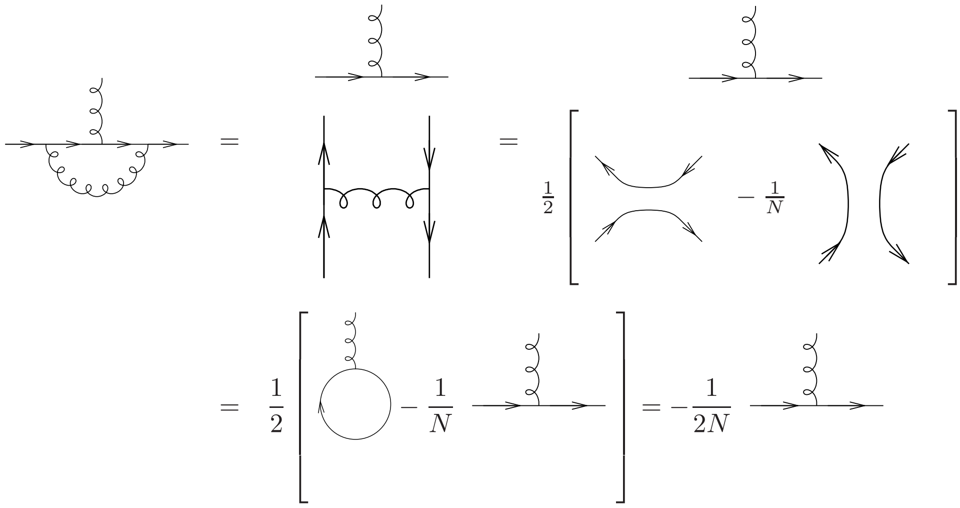
$$\boxed{\left(t^a\, t^b\, t^a\right)_{ij} = -\frac{1}{2N}\, t^b_{ij} \qquad \text{i.e.}} \qquad (17)$$

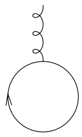


[**Solution** : Using Fierz identity one gets

$$\left(t^a\, t^b\, t^a\right)_{ij} = t^a_{ij'}\, t^b_{j'k'}\, t^a_{k'j} = \frac{1}{2}\left(\delta_{ij}\,\delta_{j'k'} - \frac{1}{N}\delta_{ij'}\,\delta_{k'j}\right) t^b_{j'k'} = -\frac{1}{2N}\, t^b_{ij}$$

since $t^b_{j'j'} = 0$ (generators are traceless).
Graphically:

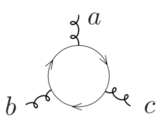
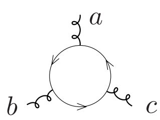


where we have used the fact that  $= \mathrm{Tr}\, t^b = 0$.]

2.2 Some color factors in adjoint representation

Graphical rule can also be given in order to compute color factor involving the adjoint representation. It requires to relate the adjoint representation to the fundamental one.

1. Prove that

$\frac{i}{2} f_{abc} =$  $-$  . (18)

[**Solution** : This graphical relation relies on the relation

$$Tr([t^a\,, t^b]\, t^c) = \frac{i}{2} f_{abc}\,. \tag{19}$$

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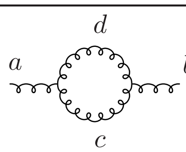
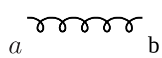
2. Derive the following identity:

$\boxed{f^{acd}\, f^{bcd} = N\,\delta^{ab} \equiv C_A\,\delta^{ab} \quad \text{i.e.}}$  $= -\,C_A$  (20)

a. Algebraically

[**Solution** : The first proof is purely algebraic:

$$\begin{aligned} \mathrm{Tr}\left([t^a\,,\,t^c]\,[t^b\,,\,t^c]\right) &= \mathrm{Tr}\left(i\,f^{acd}\,t^d\,i\,f^{bce}\,t^e\right) \\ &= -f^{acd}\,f^{bce}\,\mathrm{Tr}(t^d\,t^e) = -\frac{1}{2}\,f^{acd}\,f^{bcd}\,, \end{aligned} \tag{21}$$

thus

$$\begin{aligned} f^{acd}\,f^{bcd} &= -2\,\mathrm{Tr}\left([t^a\,,\,t^c]\,[t^b\,,\,t^c]\right) \\ &= -2\,\mathrm{Tr}\left(t^a\,t^c\,t^b\,t^c + t^c\,t^a\,t^c\,t^b - t^c\,t^a\,t^b\,t^c - t^a\,t^c\,t^c\,t^b\right) \\ &= -2\,\mathrm{Tr}\left(2\,t^a\,t^c\,t^b\,t^c - (t^a\,t^b + t^b\,t^a)\,t^c\,t^c\right) \end{aligned} \tag{22}$$

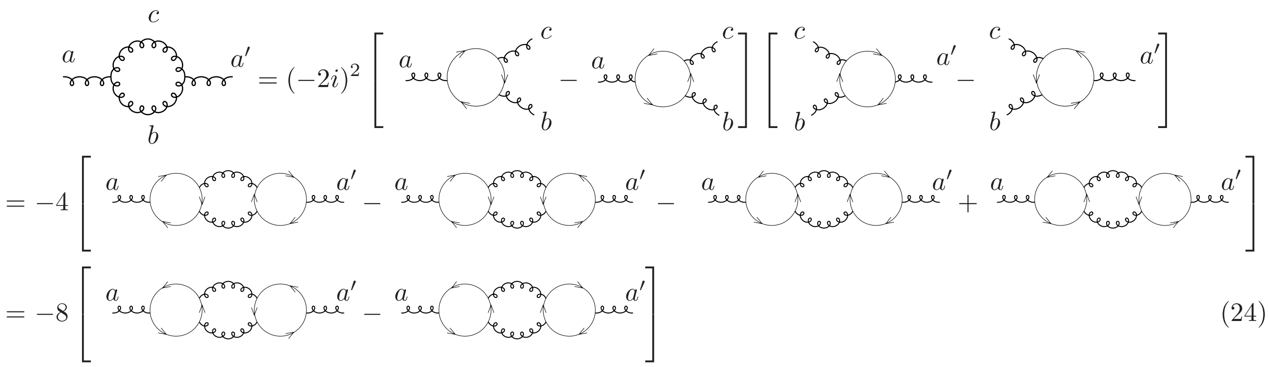
where we have used the invariance of the trace under cyclic permutation, which finally gives, using relations (16) and (17), as well as normalizations (6),

$$f^{acd}\,f^{bcd} = -2\,\mathrm{Tr}\left(-\frac{1}{N}\,t^a\,t^b - \frac{N^2-1}{2\,N}2\,t^a\,t^b\right) = 2\,N\,\mathrm{Tr}\,(t^a\,t^b) = N\,\delta^{ab}\,. \tag{23}$$

]

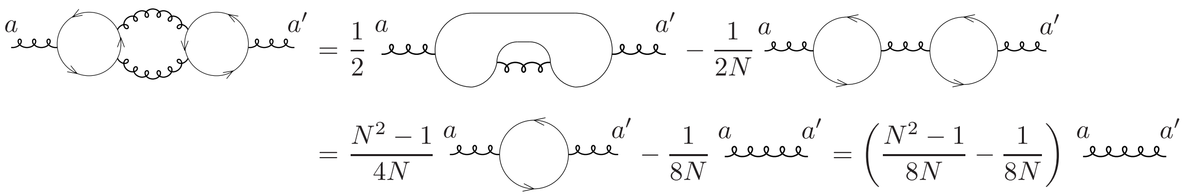
b. Graphically.

[**Solution** : From (18, we have



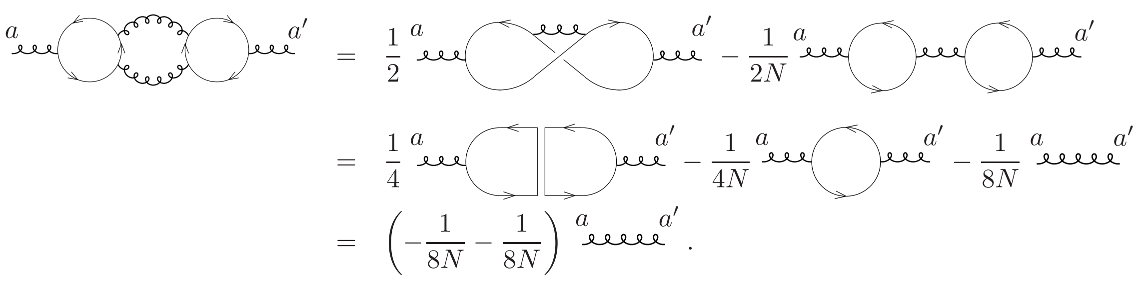
(24)

Each of these terms can now be evaluated separately using the Fierz identity:



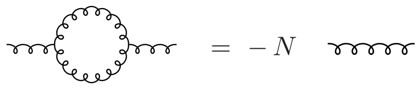
(25)

and



(26)

where we have used the fact that the first terms in the right-hand-side of the second line of the above equation vanishes since generators are traceless. Thus, inserting Eq.(25) and Eq.(26) inside Eq.(24) one gets



(27)

q.e.d.
]

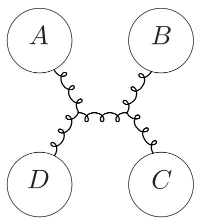
3 Color factors for the n gluon production

A tree-level multigluon amplitude can be written in an $SU(N)$ Yang-Mills theory as

$$\mathcal{M}_n = \sum_{[1,2,\cdots n]'} \mathrm{Tr}\,(t^{a_1} t^{a_2} \cdots t^{a_n})\; m(p_1, \epsilon_1; p_2, \epsilon_2; \cdots ; p_n, \epsilon_n)\,, \tag{28}$$

where $a_1, a_2, \cdots a_n\,, p_1\,, p_2\,, \cdots\,, p_n$ and $\epsilon_1\,, \epsilon_2\,, \epsilon_n$ are, respectively, the colors, momenta, and helicities of the gluons, and the sum is over the noncyclic permutations of the set $[1, 2, \cdots n]$. Our aim is to evaluate the color factors involved in the computation of the squared matrix element, averaged over colors for the initial state.

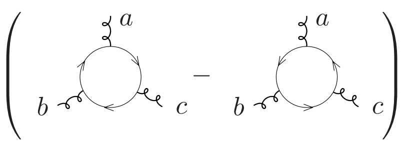
Explain why a tree-level amplitude only involves single traces.
[**Solution** : Consider a tree-like color structure, corresponding to a given Feynman diagram involved in a tree-level amplitude involving n external gluons. At tree order, the only allowed vertices are 3-gluons and 4-gluons ones (quarks are only involved at loop level). From the color point of view, a 4-gluons vertex looks like two 3-gluons vertices connected by a gluon line, so that the problem reduces to consider a pure tree-like structure build from 3-gluons vertices. It thus looks like



(29)

where A, B, C and D are themselves tree-like diagrams. First, we pass from the adjoint representation to the fundamental one by using the identity (18). This identity can be rewritten as

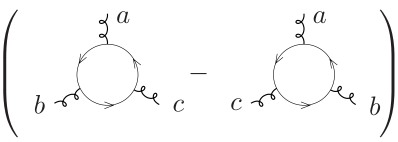
$f_{abc} = -2i$



(30)

i.e.

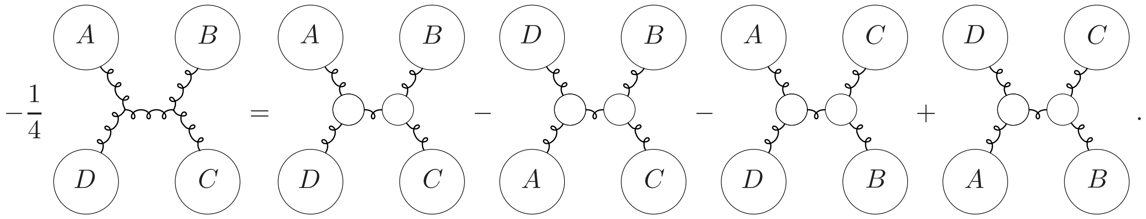
$f_{abc} = 2i$



. (31)

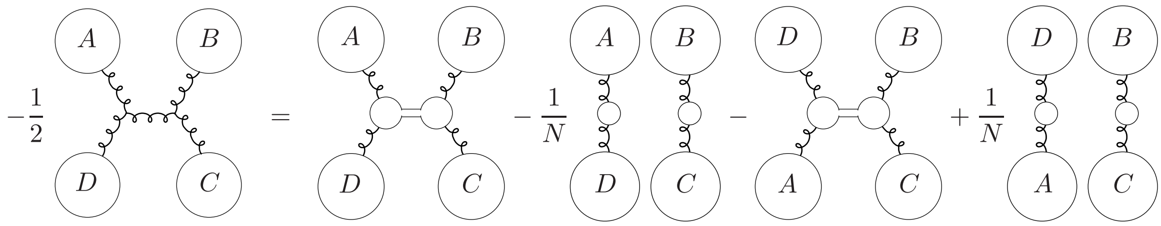
For simplicity, we now remove the arrows on the fundamental line, taking the convention that loop in the fundamental representation represent counterclockwise color flows.

The diagram (29) thus reads

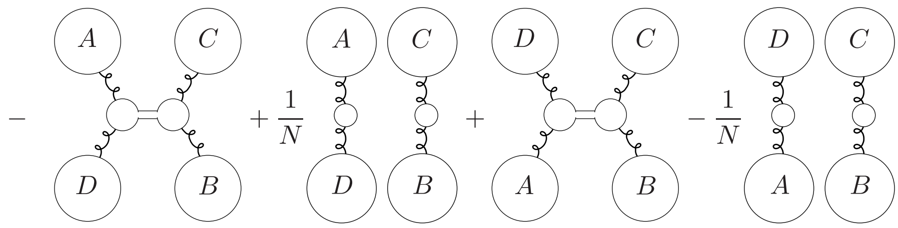


(32)

Let us apply Fierz identity to the gluon which connects the two fundamental loops. This leads to

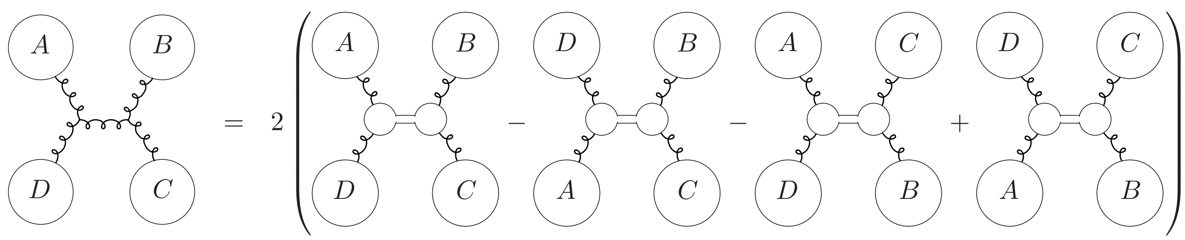


(33)



(34)

so that



(35)

Thus, using a recursive proof ending when reaching the shelves of the tree, it is clear that the whole structure can be reduced to a single trace of the product of the n generators in the fundamental representation, the various terms being the trace $\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})$ and their non-cyclic permutations.]

3.1 A particular case

We first consider the color factor associated to $|\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})\,|^2$. We denote the corresponding color factor as

$$C_n = \frac{1}{(N^2-1)^2}|\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})\,|^2 \equiv \frac{1}{(N^2-1)^2}T_n\,. \tag{36}$$

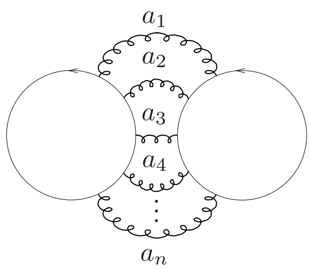
[**Solution** : As a preliminary step, let us study the structure of $|\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})\,|^2$. It reads

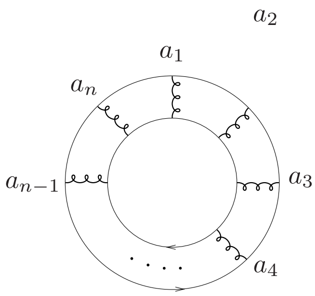
$$[\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})]^* = \mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})^\dagger = \mathrm{Tr}\,(t^{a_n}t^{a_2}\cdots t^{a_1}) \tag{37}$$

where we have used the fact that t^{a_i} are hermitian. Thus,

$$|\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})\,|^2 = \mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})\ \mathrm{Tr}\,(t^{a_n}t^{a_2}\cdots t^{a_1}) \tag{38}$$

which reads graphically

$$|\mathrm{Tr}\,(t^{a_1}t^{a_2}\cdots t^{a_n})\,|^2 = $$ 

$$=$$ 

(39)

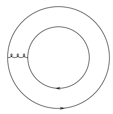
the second equality being trivial by an obvious deformation.]
1. Show that

$$T_1 = 0\,, \tag{40}$$

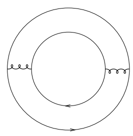
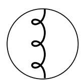
$$T_2 = \frac{N^2-1}{4}\,, \tag{41}$$

$$T_3 = \frac{(N^2-2)(N^2-1)}{8N}\,. \tag{42}$$

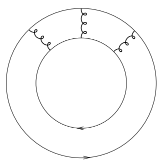
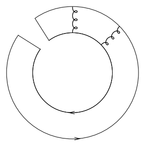
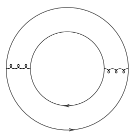
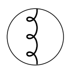
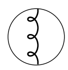
[**Solution** : Obviously,

$$T_1 = $$  $$= 0\,.$$

Next, from Eq. (15),

$$T_2 = $$  $$= \frac{1}{2}$$  $$= \frac{N^2-1}{4}\,.$$

Finally,

$$T_3 = $$  $$= \frac{1}{2}$$  $$- \frac{1}{2N}$$  $$= \frac{C_F}{2}$$  $$- \frac{1}{2N}\frac{1}{2}$$  $$= \left(\frac{N^2-1}{4N} - \frac{1}{4N}\right)\frac{N^2-1}{2}\,,$$

so that

$$T_3 = \frac{(N^2-2)(N^2-1)}{8N}\,.$$

]
2. Justify the fact that

$$T_n \sim \left(\frac{N}{2}\right)^n \tag{43}$$

for $N \to \infty$ ('t Hooft limit).

[**Solution** : One can apply the Fierz identity as we did above for T_2 and T_3. In the large N limit, only the first term in Eq. (10) remains. Each gluon thus contributes with a factor of $N/2$, leading to

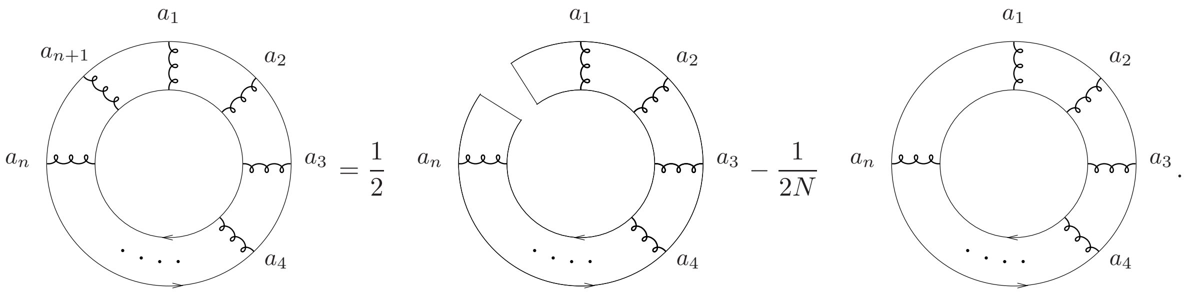
$$T_n \sim \left(\frac{N}{2}\right)^n\,.$$

]

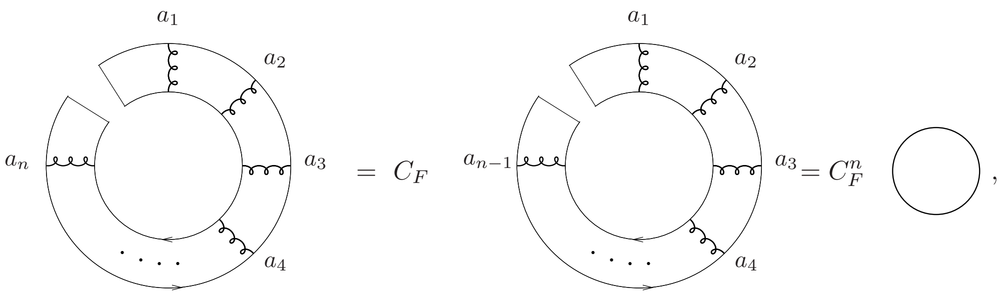
3. Prove that

$$T_{n+1} = \frac{N}{2}\, C_F^n - \frac{1}{2N} T_n \,. \tag{44}$$

[**Solution** : The proof relies on the use of the Fierz identity (10), which gives



Now



thus leading to Eq. (44).]

4. Solve the relation (44) and show that

$$T_n = \frac{N^2-1}{N^n} \frac{1}{(-2)^n} \left[1 - (1-N^2)^{n-1}\right] \,. \tag{45}$$

[**Solution** : Introducing

$$A_n = \frac{(-2N)^n}{N^2}\, T_n \,,$$

the recursion (44) reads

$$A_{n+1} = -K^n + A_n \,,$$

with $K = -2N\,C_F = 1 - N^2$. This is solved by

$$A_n = -K \frac{1-K^{n-1}}{1-K} \,,$$

i.e.

$$A_n = \frac{(N^2-1)}{N^2} (1 - (1-N^2)^{n-1}) \,.$$

Thus

$$T_n = \frac{N^{2-n}}{(-2)^n} A_n = \frac{N^2-1}{N^n} \frac{1}{(-2)^n} \left[1 - (1-N^2)^{n-1}\right] \,,$$

which satisfies, as expected, $T_n \sim \left(\frac{N}{2}\right)^n$.]

3.2 The planar limit

1. For a few number of gluons, evaluate the other color factors occuring when squaring the various matrix elements involved in Eq. (28).

For simplicity, we now replace $SU(N)$ by $U(N)$, and we restrict ourselves to the $N \to \infty$ limit. In 1974 (Nuclear Physics B72 (1974) 461-473), 't Hooft proved that one can make a simple evaluation of the scaling of a given diagram in this $U(N)$ Yang-Mills theory by a simple investigation of the two-dimensional surface in one-to-one correspondence with this diagram. The physical motivation to do so is to introduce a new expansion parameter, allowing for a systematic classification of diagrams, besides the usual coupling g involved in perturbative methods.

We will not give here the detailled Feynman rules for the $U(N)$ Yang-Mills theory. The only thing which we need is the fact that quarks (antiquark) live in the fundamental representation N ($\bar{N}$) of $U(N)$, while the gluon are in the adjoint representation. The various propagator thus carry color indices in accordance with these representations, while the vertices involve the generators in fundamental (for the quark-gluon vertex) and adjoint representations (for 3-gluons, 4-gluons vertices and gluon-ghost vertex). This is illustrated in Fig. 1 (for simplicity we do not consider 4-gluons and ghosts vertices).

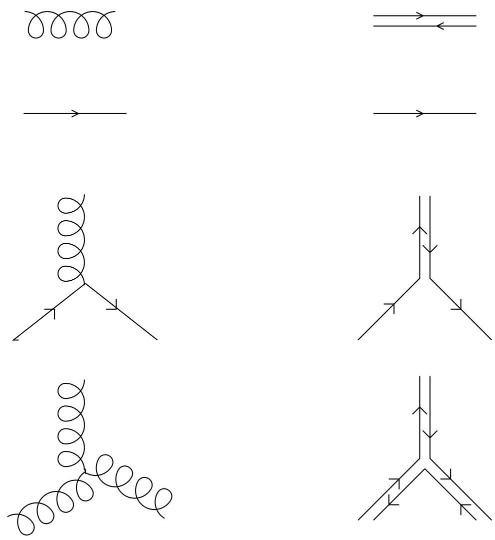


Figure 1: 't Hooft representation for a $U(N)$ gauge theory.

To make a mapping between the color structure of a given Feynman diagram and a two-dimensional surface, one should attach little surfaces to each color index loop. This leads to a big surface, with edges formed by the quark lines. This surface is in general multiply connected: it contains "worm holes" (for example because of in internal quark loop). The surface can be closed by attaching little surfaces to the quark loops separately.

2. Find the surface to be drawn for the color structures investigated in section 3.1.

3. Find the surfaces corresponding to the other color factors occuring in Eq. (28), for the case of $n = 2$ and $n = 3$ gluons. Can you identify the difference between the type of surfaces which are involved?

4. Justify the fact that the structures investigated in section 3.1 are the dominant one for a given value of n. What is the topology of the corresponding surfaces?