M2/CFP/Parcours of Physique Théorique

Invariances in physics and group theory

Exercises on Cartan algebra, roots, weights...

Equations refer to the lecture of Jean-Bernard Zuber, Chapter 3, version 2013.

A. *Cartan algebra and roots*

1. Show that any element X of $\mathfrak{g}$ may be written as $X = \sum x^i H_i + \sum_{\alpha\in\Delta} x^\alpha E_\alpha$ with the notations of § 3.1.2.

For an arbitrary H in the Cartan algebra, determine the action of $\mathrm{ad}(H)$ on such a vector X ; conclude that $\mathrm{ad}(H)\mathrm{ad}(H')X = \sum_{\alpha\in\Delta} x^\alpha \alpha(H)\alpha(H')E_\alpha$ and taking into account that the eigenspace of each root α has dimension 1, cf point $(*)$ 2. of § 3.1.2, that the Killing form reads

$$(H, H') = \mathrm{tr}(\mathrm{ad(H)ad(H')}) = \sum_{\alpha\in\Delta} \alpha(\mathrm{H})\alpha(\mathrm{H}') \ . \tag{1}$$

[**Solution** : For a fixed value of j, H_j has $\dim\mathfrak{g}$ independent eigenvectors (since H_j acts on $\mathfrak{g}$). These eigenvectors thus provide a set of generators, allowing to write

$$\forall X \in \mathfrak{g}, X = \sum x^i H_i + \sum_{\alpha\in\Delta} x^\alpha E_\alpha \, . \tag{2}$$

For $H \in \mathfrak{h}$,

$$\mathrm{ad}H\,X = [H, X] = \sum_{\alpha\in\Delta} x^\alpha [H, E_\alpha] = \sum_{\alpha\in\Delta} x^\alpha \alpha(H) E_\alpha \, .$$

Thus

$$\mathrm{ad}(H)\,\mathrm{ad}(H')\,X = \sum_{\alpha\in\Delta} x^\alpha \alpha(H')\,[H, E_\alpha] = \sum_{\alpha\in\Delta} x^\alpha \alpha(H')\,\alpha(H)\,E_\alpha$$

and

$$\alpha(H')\,\alpha(H)\,E_\alpha = \alpha(H')\,\alpha(H)\,E_\alpha$$

for a given α . Each root α corresponds to an eigenspace spanned by E_α, of dimension 1. Thus

$$\mathrm{tr}[\mathrm{ad(H)}\,\mathrm{ad(H')}] = \sum_{\alpha\in\Delta} \alpha(\mathrm{H'})\,\alpha(\mathrm{H})\,.$$

]

2. One wants to show that roots α defined by (3.5) or (3.6) generate all the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$. Prove that if it were not so, there would exist an element H of $\mathfrak{h}$ such that

$$\forall\alpha\in\Delta \quad \alpha(H)=0\ . \tag{3}$$

Using (1) show that this would imply $\forall H'\in\mathfrak{h}$, $(H,H')=0$. Why is that impossible in a semi-simple algebra? (see the discussion before equation (3.10)).

[**Solution** : One is looking for $H=\sum_i h^i\,H_i\in\mathfrak{h}$, so that $\alpha(H)=\sum_i h^i\,\alpha_{(i)}=0$ for every $\alpha\in\Delta$. Thus, H should obey $|\Delta|$ equations, which would lead to a number of undetermined parameters larger or equal to $\dim\mathfrak{h}-|\Delta|$, $\{h^i,\, i\in\{1\cdots\dim\mathfrak{h}\}\}$ being a solution of this set of equations. In the case where $|\Delta|<\dim\mathfrak{h}^*=\dim\mathfrak{h}$, a non-trivial solution (i.e. such that h^i do not all vanish) thus exists, and H is non zero.

The expression

$$(H,\,H')=\sum_{\alpha\in\Delta}\alpha(H')\,\alpha(H)$$

leads to

$$\forall H'\in\mathfrak{h}\,,(H,\,H')=0\,.$$

Since $(H,\,E_\alpha)=0$, we thus deduce that

$$\forall X\in\mathfrak{h}\,,(H,\,X)=0\,,$$

thus showing that the Killing form is degenerate on $\mathfrak{h}$, in contradiction with the semi-simplicity of $\mathfrak{h}$. Thus, Δ spans $\mathfrak{h}^*$ (and thus $|\Delta|\geq\dim\mathfrak{h}^*=\dim h$).]

3. Variant of the previous argument : under the assumption of 2. and thus of (3), show that H would commute with all H_i and all the E_α, thus would belong to the center of $\mathfrak{g}$. Prove that the center of an algebra is an abelian ideal. Conclude in the case of a semi-simple algebra.

[**Solution** : Assume that $\exists H \in \mathfrak{h}$ such that $\forall \alpha \in \Delta\,, \alpha(H) = 0\,$. Then, on one hand $[H,\, H_i] = 0\,$, while on the other hand $\forall \alpha \in \Delta$, $[H\,, E_\alpha] = \alpha(H)\, E_\alpha = 0$ so that $H \in Z_{\mathfrak{g}}\,$. Now, $X \in Z_{\mathfrak{g}} \Leftrightarrow \forall\, Y \in \mathfrak{g}, [X, Y] = 0\,$. Obviously, $[Z_{\mathfrak{g}}, \mathfrak{g}] \subset Z_{\mathfrak{g}}$ since $[Z_{\mathfrak{g}}, \mathfrak{g}] = \{0\}$, showing that $Z_{\mathfrak{g}}$ is an abelian ideal of $\mathfrak{g}$ (abelian since every element of $Z_{\mathfrak{g}}$ commute with every element of $\mathfrak{g}$, in particular those of $Z_{\mathfrak{g}}$). Based on the semi-simplicity of $\mathfrak{g}$, this implies that $Z_{\mathfrak{g}} = \{0\}$, proving that $H \in Z_{\mathfrak{g}}$ is impossible.]

B. *Computation of the* $N_{\alpha\beta}$

1. Show that the real constants $N_{\alpha\beta}$ satisfy $N_{\alpha\beta} = -N_{\beta\alpha}$ and, by complex conjugation of $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$ that

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}\ . \tag{4}$$

[**Solution** : $[E_\alpha, E_\beta] = N_{\alpha,\beta}\, E_{\alpha+\beta}$ if $\alpha + \beta$ is a root. Since $[E_\alpha, E_\beta] = -[E_\beta, E_\alpha]$, we thus have $N_{\beta,\alpha} = -N_{\alpha,\beta}\,$.
$[H, E_\alpha] = \alpha\, E_\alpha$ leads to $[H, E_\alpha]^\dagger = -[H, E_\alpha^\dagger] = \alpha\, E_\alpha^\dagger$ since H is hermitian. Thus, $[H, E_\alpha^\dagger] = -\alpha\, E_\alpha^\dagger$ and thus $E_\alpha^\dagger = E_{-\alpha}\,$. From $[E_\alpha, E_\beta] = N_{\alpha,\beta}\, E_{\alpha+\beta}$ we thus have $[E_\alpha, E_\beta]^\dagger = N_{\alpha,\beta}\, E_{\alpha+\beta}^\dagger = N_{\alpha,\beta}\, E_{-\alpha-\beta}\,$. This leads to

$$\boxed{N_{\alpha,\beta} = N_{-\beta,-\alpha} = -N_{-\alpha,-\beta} = -N_{\beta,\alpha}\,.}$$

]

2. Consider three roots satisfying $\alpha + \beta + \gamma = 0$. Writing the Jacobi identity for the triplet E_α, E_β, E_γ, show that $\alpha_{(i)} N_{\beta\gamma} + \text{cycl.} = 0$. Derive from it the relation

$$N_{\alpha\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}\ . \tag{5}$$

[**Solution** : The Jacobi identity reads

$$[[E_\alpha, E_\beta], E_\gamma] + [[E_\beta, E_\gamma], E_\alpha] + [[E_\gamma, E_\alpha], E_\beta] = 0$$

and thus

$$N_{\alpha,\beta}\,[E_{\alpha+\beta}, E_\gamma] + N_{\beta,\gamma}\,[E_{\beta+\gamma}, E_\alpha] + N_{\gamma,\alpha}\,[E_{\alpha+\gamma}, E_\beta] = 0\,,$$

which can be written as

$$N_{\alpha,\beta}\,[E_{-\gamma}, E_\gamma] + N_{\beta,\gamma}\,[E_{-\alpha}, E_\alpha] + N_{\gamma,\alpha}\,[E_{-\beta}, E_\beta] = 0\,,$$

i.e.

$$N_{\alpha,\beta}\, H_{-\gamma} + N_{\beta,\gamma}\, H_{-\alpha} + N_{\gamma,\alpha}\, H_{-\beta} = 0\,,$$

so that

$$N_{\beta,\gamma}\,(-\alpha_{(j)} H_j) + \text{cycl.perm.} = 0\,.$$

Calculating $(H_i,\ )$ for a fixed value of i, we thus get

$$\boxed{\alpha_{(i)}\, N_{\beta,\gamma} + \text{cycl.perm.} = 0\,.}$$

We thus have

$$\alpha\, N_{\beta,\gamma} + \beta\, N_{\gamma,\alpha} + \gamma\, N_{\alpha,\beta} = 0$$

so that

$$\alpha(N_{\beta,-\alpha-\beta} - N_{\alpha,\beta}) + \beta(N_{-\alpha-\beta,\alpha} - N_{\alpha,\beta}) = 0\,.$$

This identity is valid for any set of roots α, β, γ satisfying $\alpha + \beta + \gamma = 0$, so that α and β are linearly independent, except if $\beta = \pm\alpha$. Now :

• If $\alpha = \beta$, then $\gamma = -2\alpha$ which is impossible since the only roots of the form $\lambda\,\alpha$ are $\pm\alpha$.

• If $\alpha = -\beta$, then $\gamma = 0$ which is again impossible.

We thus conclude, based on the linear independence of α and β, that

$$\boxed{N_{\beta,-\alpha-\beta} = N_{\alpha,\beta} = N_{-\alpha-\beta,\alpha}\,.}$$

This identity remains trivially satisfied for $\beta = \alpha$, since on one hand $[E_\alpha, E_\alpha] = 0$, i.e. $N_{\alpha,\alpha} = 0$ and on the other hand $N_{\alpha,-2\alpha} = 0$ since -2α is not a root. Similarly, it is also valid in the case $\beta = -\alpha$ since $N_{\alpha,-\alpha} = N_{-\alpha,0} = N_{0,\alpha} = 0$. The above identity is thus valid for any set of roots α and β.]

3. Considering the α-chain through β and the two integers p et q defined in § 3.2.1, write the Jacobi identity for E_α, $E_{-\alpha}$ and $E_{\beta+k\alpha}$, with $p \le k \le q$, and show that it implies

$$\langle \alpha, \beta + k\alpha \rangle = N_{\alpha,\beta+k\alpha} N_{\alpha,\beta+(k-1)\alpha} + N_{\beta+k\alpha,\alpha} N_{-\alpha,\beta+(k+1)\alpha}\ .$$

Let $f(k) := N_{\alpha,\beta+k\alpha} N_{-\alpha,-\beta-k\alpha}$. Using the relations (5), show that the previous equation may be recast as

$$\langle \alpha, \beta + k\alpha \rangle = f(k) - f(k-1)\ . \tag{6}$$

[**Solution** : The Jacobi identity reads

$$[[E_\alpha, E_{-\alpha}], E_{\beta+k\alpha}] + [[E_{-\alpha}, E_{\beta+k\alpha}], E_\alpha] + [[E_{\beta+k\alpha}], E_\alpha], E_{-\alpha}] = 0\,,$$

i.e.

$$[H_\alpha, E_{\beta+k\alpha}] + N_{-\alpha,\beta+k\alpha}\,[E_{\beta+(k-1)\alpha}, E_\alpha] + N_{\beta+k\alpha,\alpha}\,[E_{\beta+(k+1)\alpha}, E_{-\alpha}] = 0\,,$$

and thus

$$\sum_i \alpha(i)[\beta(i) + k\,\alpha(i)]\,E_{\beta+k\alpha} + N_{-\alpha,\beta+k\alpha}\,N_{\beta+(k-1)\alpha,\alpha}\,E_{\beta+k\alpha} \\ + N_{\beta+k\alpha,\alpha}\,N_{\beta+(k+1)\alpha,-\alpha}\,E_{\beta+k\alpha} = 0$$

From $\sum_i \alpha(i)[\beta(i)+k\,\alpha(i)] = \langle\alpha, \beta+k\,\alpha\rangle$, $N_{\beta+(k-1)\alpha,\alpha} = -N_{\alpha,\beta+(k-1)\alpha}$ and $N_{\beta+(k+1)\alpha,-\alpha} = -N_{-\alpha,\beta+(k+1)\alpha}$ we get, since $E_{\beta+k\alpha} \neq 0$,

$$\boxed{\langle\alpha, \beta + k\,\alpha\rangle = N_{-\alpha,\beta+k\alpha}\,N_{\alpha,\beta+(k-1)\alpha} + N_{\beta+k\alpha,\alpha}\,N_{-\alpha,\beta+(k+1)\alpha}\,.}$$

From the definition of f, we have

$$f(k) = N_{\alpha,\beta+k\alpha}N_{-\alpha,-\beta-k\alpha} = -N^2_{\alpha,\beta+k\alpha} = N_{\beta+k\alpha,\alpha}N_{-\beta-k\alpha,-\alpha} = N_{\beta+k\alpha,\alpha}N_{-\alpha,\beta+(k+1)\alpha}\,.$$

Thus,

$$f(k-1) = N_{\alpha,\beta+(k-1)\alpha}N_{-\alpha,-\beta-(k-1)\alpha}\,.$$

Now, since $N_{-\alpha,-\beta-(k-1)\alpha} = N_{\beta+k\alpha,-\alpha} = -N_{-\alpha,\beta+k\alpha}$, where we have used $N_{\gamma,\delta} = -N_{-\gamma-\delta,\gamma}$, we thus have

$$\boxed{\langle\alpha, \beta + k\,\alpha\rangle = f(k) - f(k-1)\,.}$$

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4. What are $f(q)$ et $f(q-1)$? Show that the recursion relation (6) is solved by

$$f(k) = -(N_{\alpha,\beta+k\alpha})^2 = (k-q)\langle\alpha, \beta + \frac{1}{2}(k+q+1)\rangle\ . \tag{7}$$

What is $f(p-1)$? Show that the expression (3.21) is recovered. Show that (7) is in accord with (3.23). The sign of the square root is still to be determined . . ., see [Gi].

[**Solution** : The definition of f leads to $f(q) = N_{\alpha,\beta+q\alpha}N_{-\alpha,-\beta-q\alpha}$. Since $[E_\alpha, E_{\beta+q\alpha}] = N_{\alpha,\beta+q\alpha}E_{\beta+(q+1)\alpha}$ while $\beta + (q+1)\alpha$ is not root, as implied by the definition of q, we obtain $N_{\alpha,\beta+q\alpha} = 0$ and thus

$$\boxed{f(q) = 0\,.}$$

From the definition of f,

$$f(q-1) = -N^2_{\alpha,\beta+(q-1)\alpha}\,.$$

On the other hand, $\langle\alpha,\beta+q\,\alpha\rangle = f(q) - f(q-1)$, so that

$$\boxed{f(q-1) = -\langle\alpha,\beta+q\,\alpha\rangle = -N^2_{\alpha,\beta+(q-1)\alpha}\,.}$$

From the expression $\langle\alpha,\beta\rangle + k\langle\alpha,\alpha\rangle = f(k) - f(k-1)$ we get

$$f(q)-f(q-1)+f(q-1)-f(q-2)+\cdots+f(k+1)-f(k) = -f(k) = (q-k)\langle\alpha,\beta\rangle + \sum_{m=k+1}^{q} m\langle\alpha,\alpha\rangle\,.$$

From $\sum\limits_{m=k+1}^{q} m = -\sum\limits_{m=1}^{k} m + \sum\limits_{m=1}^{q} = \frac{q(q+1)}{2} - \frac{k(k+1)}{2} = -\frac{k-q}{2}(1+k+q)$ we get

$$f(k) = (k-q)\langle\alpha,\beta\rangle + \frac{1}{2}\langle\alpha,\alpha\rangle(1+k+q)(k-q)$$

so that

$$\boxed{f(k) = (k-q)\langle\alpha,\beta+\tfrac{1}{2}(k+q+1)\alpha\rangle = -N^2_{\alpha,\beta+k\alpha}\,.}$$

We now compute

$$f(p-1) = N_{\alpha,\beta+(p-1)\alpha}N_{-\alpha,-\beta-(p-1)\alpha}\,.$$

The second coefficient reads $N_{-\alpha,-\beta-(p-1)\alpha} = N_{\beta+p\alpha,-\alpha} = -N_{-\alpha,\beta+p\alpha}$. Since

$$[E_{-\alpha},E_{\beta+p\alpha}] = N_{-\alpha,\beta+p\alpha}E_{\beta+(p-1)\alpha}$$

and using the fact that $\beta+(p-1)\alpha$ is not a root by definition of q, we deduce that $N_{-\alpha,\beta+p\alpha} = 0$ and thus $f(p-1) = 0$. Therefore,

$$f(p) = \langle\alpha,\beta+p\alpha\rangle = -N^2_{\alpha,\beta+p\alpha} = (p-q)\langle\alpha,\beta+\frac{1}{2}(p+q+1)\alpha\rangle\,,$$

so that

$$\langle\alpha,\beta\rangle(p-q-1) = [p-\frac{1}{2}(p-q)(p+q+1)]\langle\alpha,\alpha\rangle = -\frac{1}{2}(p-q-1)(p+q)\langle\alpha,\alpha\rangle\,.$$

Therefore,

$$2\langle\alpha,\beta\rangle = (-p-q)\langle\alpha,\alpha\rangle = m\langle\alpha,\alpha\rangle\,.$$

We thus have

$$\boxed{2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = m\in\mathbb{Z}\,.}$$

We finally get

$$f(0) = -N^2_{\alpha,\beta} = -q\langle\alpha, \beta + \frac{1}{2}(q+1)\alpha\rangle = -q[\langle\alpha,\beta\rangle + \frac{1}{2}(q+1)\langle\alpha,\alpha\rangle] = -\frac{q(1-p)}{2}\langle\alpha,\alpha\rangle$$

and thus

$$\boxed{|N_{\alpha,\beta}| = \sqrt{\frac{q(1-p)}{2}\langle\alpha,\alpha\rangle}\,.}$$

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C. *Study of the* B_l =$so(2l+1)$ *and* G_2 *algebras*

1. $\underline{so(2l+1) = B_l}$, $l \geq 2$

a. What is the dimension of the group SO$(2l+1)$ or of its Lie algebra so$(2l+1)$?

[**Solution** : We start from the defining relation

$$O\,O^t = O^t\,O = 1$$

with O expanded as $O = 1 + iT$, i.e. $(1+iT)(1+iT^t) = 1$ and thus $i(T + T^t) = 0$, showing that T is antisymmetric.

If T is an $n \times n$ matrix (for the case of $O(n)$ or $SO(n)$), T is characterized by $(n^2 - n)/2$ elements, so that for $n = 2\ell + 1$,

$$\boxed{\dim o(2\ell+1) = \ell(2\ell+1)\,.}$$

This dimension is the same for $so(2\ell+1)$ since $\det O = 1$ does not introduce any additional constraint on the Lie algebra (it only selects the connected component of the identity when considering the group).]

b. What is the rank of the algebra ? (Hint : diagonalize a matrix of so$(2l+1)$ on $\mathbb{C}$, or write it as a diagonal of real 2×2 blocks, see lecture notes, §3.1)

[**Solution** : Since O is diagonalizable in the form

$$D = \mathrm{diag}\left(0, \begin{pmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{pmatrix}_{j=1,\cdots,\ell}\right)$$

with real μ_j (see chapter 3), we thus get

$$\boxed{\mathrm{rank}\, so(2\ell+1) = \ell\,.}$$

]

c. How many roots does the algebra have ? How many positive roots ? How many simple ?

[**Solution** : # of roots = dimension - rank $= \ell(2\ell+1) - \ell = 2l^2$.
Among these roots, half of them are positive, i.e. ℓ^2. The number of simple roots is equal to the rank, i.e. ℓ .]

d. Let e_i, $i = 1, \cdots, l$ be a orthonormal basis in $\mathbb{R}^l$, $\langle e_i, e_j\rangle = \delta_{ij}$. Consider the set of vectors

$$\Delta = \{\pm e_i \ , \ 1 \le i \le l\} \cup \{\pm e_i \pm e_j \ , \quad 1 \le i < j \le l\}$$

What is the cardinal of Δ ?

[**Solution** : Card $\{\pm e_i \ , \ 1 \le i \le l\} = 2\ell$ and Card $\{\pm e_i \pm e_j \ , \quad 1 \le i < j \le l\} = 4C_\ell^2 = 2l(l-1)$, therefore Card $\Delta = 2l^2$.]

Δ is the set of roots of the algebra so$(2l+1)$.

e. A basis of simple roots is given $\alpha_i = e_i - e_{i+1}$, $i = 1, \cdots, l-1$, et $\alpha_l = e_l$. Explain why the roots

$$\begin{aligned} e_i &= \sum_{i \le k \le l} \alpha_k, \quad 1 \le i \le l\,, \\ e_i - e_j &= \sum_{i \le k < j} \alpha_k, \quad 1 \le i < j \le l\,, \\ e_i + e_j &= \sum_{i \le k < j} \alpha_k + 2\sum_{j \le k \le l} \alpha_k, \quad 1 \le i < j \le l\,, \end{aligned} \tag{8}$$

qualify as positive roots.

[**Solution** : There are

$$\begin{aligned} e_i &= \sum_{i \le k \le l} \alpha_k, & 1 \le i \le l\,, && \text{Card} = \ell\,, \\ e_i - e_j &= \sum_{i \le k < j} \alpha_k, & 1 \le i < j \le l\,, && \text{Card} = \frac{\ell(\ell-1)}{2}\,, \\ e_i + e_j &= \sum_{i \le k < j} \alpha_k + 2\sum_{j \le k \le l} \alpha_k, & 1 \le i < j \le l\,, && \text{Card} = \frac{\ell(\ell-1)}{2}\,, \end{aligned} \tag{9}$$

there are thus $|\Delta_+| = l^2$ of such roots, which expand on the set of simple roots with integer coefficients. Including their opposite (negative roots), they correctly reproduce the whole set Δ.] Check that assertion on the case of $B_2 = so(5)$.

[**Solution** : For $B_2 = so(5)$, $\Delta = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ with Card $\Delta = 8$. The $\ell = 2$ (which is equal to the rank of $B_2 = so(5)$) simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$. And $\Delta_+ = \{e_1, e_2, e_1 - e_2, e_1 + e_2\}$ with Card $\Delta_+ = 4$.]

f. Compute the Cartan matrix and check that it agrees with the Dynkin diagram given in the notes.

[**Solution** : Obviously, $2\frac{\langle\alpha_i,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} = 2$. Then,

$$2\frac{\langle\alpha_{j+1},\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = 2\frac{\langle e_{j+1}-e_{j+2},e_j-e_{j+1}\rangle}{\langle e_j-e_{j+1},e_j-e_{j+1}\rangle} = \frac{2(-1)}{2} = -1\,,$$

and

$$2\frac{\langle\alpha_j,\alpha_{j+1}\rangle}{\langle\alpha_{j+1},\alpha_{j+1}\rangle} = -1\,.$$

Finally,

$$2\frac{\langle\alpha_{\ell-1},\alpha_\ell\rangle}{\langle\alpha_\ell,\alpha_\ell\rangle} = 2\frac{\langle e_{\ell-1}-e_\ell,e_\ell\rangle}{\langle e_\ell,e_\ell\rangle} = -2\,,$$

and

$$2\frac{\langle\alpha_\ell,\alpha_{\ell-1}\rangle}{\langle\alpha_{\ell-1},\alpha_{\ell-1}\rangle} = 2\frac{\langle e_\ell,e_{\ell-1}-e_\ell\rangle}{\langle e_{\ell-1}-e_\ell,e_{\ell-1}-e_\ell\rangle} = \frac{2(-1)}{2} = -1\,,$$

which can be summarized by

$$C_{ij} = 2\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = \begin{cases} 2 & \text{si } 1\le i=j\le l \\ -1 & \text{si } 1\le i=(j\pm 1)\le l-1 \\ -2 & \text{si } i=l-1\,, j=l \\ -1 & \text{si } i=l\,, j=l-1 \end{cases}$$

corresponding to the Dynkin diagram illustrated in Fig. 1. The roots of B_2 are

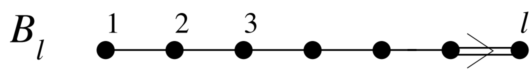

FIGURE 1 – The Dynkin diagram for the algebra B_ℓ.

illustrated in Fig. 2]

g. Compute the sum ρ of positive roots.

[**Solution** :

$$2\rho = \sum_{i=1}^{\ell} e_i + \sum_{1\le i<j\le\ell}(e_i-e_j+e_i+e_j) = \sum_{i=1}^{\ell} e_i + 2\sum_{1\le i<j\le\ell} e_i = \sum_{1\le i<j\le\ell}[1+2(\ell-i)]e_i$$
$$= (2l-1)e_1+(2l-3)e_2+\cdots+(2l-2i+1)e_i+\cdots+3\,e_{l-1}+e_l$$
$$= (2l-1)(\alpha_1+\cdots\alpha_\ell)+(2l-3)(\alpha_2+\cdots\alpha_\ell)+\cdots+(2l-2i+1)(\alpha_i+\cdots\alpha_\ell)$$
$$+\cdots+3\,(\alpha_{\ell-1}+\alpha_\ell)+\alpha_l$$
$$= (2l-1)\alpha_1+2(2l-2)\alpha_2+\cdots+i(2l-i)\alpha_i+\cdots+l^2\alpha_l\,.$$

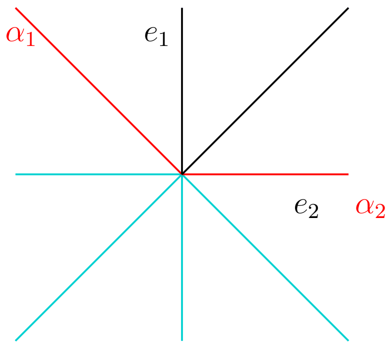

FIGURE 2 – The roots of B_2. In red, the two simple roots α_1 and α_2. In black, the two other positive roots, and in cyan the four negative roots.

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h. The Weyl is the ("semi-direct") product $W \equiv \mathcal{S}_l \ltimes (\mathbb{Z}_2)^l$, which acts on the e_i (hence on weights and roots) by permutation and by independent sign changes $e_i \mapsto (\pm 1)_i e_i$. What is its order ? In the case of B_2, check that assertion and draw the first Weyl chamber.

[**Solution** : $|W_\ell| = 2^l.l!$, and for B_2 , $|W_2| = 8$. Permutations and change of sign of e_1 and e_2 indeed corresponds to (products of) symmetries in "planes" orthogonal to α_1 and α_2 ; they do not modify the picture of Fig. 2. It is worth obtaining W_2 in a pedestrian way. We easily get

$$\begin{cases} \alpha_1^\perp e_1 = e_2 \,, & \alpha_2^\perp e_1 = e_1 \,, \\ \alpha_1^\perp e_2 = e_1 \,, & \alpha_2^\perp e_2 = -e_2 \,. \end{cases}$$

Thus

$$\begin{cases} \alpha_1^\perp \alpha_2^\perp e_1 = e_2 \,, & \alpha_2^\perp \alpha_1^\perp e_1 = -e_2 \,, \\ \alpha_1^\perp \alpha_2^\perp e_2 = -e_1 \,, & \alpha_2^\perp \alpha_1^\perp e_2 = e_1 \,. \end{cases}$$

This shows that $\alpha_1^\perp \alpha_2^\perp = -\alpha_2^\perp \alpha_1^\perp$ is a rotation of angle $-\pi/2$ (in the oriented plane e_1, e_2). Thus, any word made of $\mathbb{I}, -\mathbb{I}, \alpha_1^\perp, \alpha_2^\perp$ reduces to the elements of $W_2 = \{\mathbb{I}\,, \alpha_1^\perp\,, \alpha_2^\perp\,, \alpha_1^\perp \alpha_2^\perp\,, -\mathbb{I}\,, -\alpha_1^\perp\,, -\alpha_2^\perp\,, -\alpha_1^\perp \alpha_2^\perp\}$. Equivalently, the Weyl group is generated by the set of reflections in the hyperplane (here line) orthogonal to the various α's. The first Weyl chamber $\mathcal{C}_1 = \{\lambda | \langle \lambda, \alpha_i \rangle \geq 0\}$ is the octant between $\alpha_1 + \alpha_2$ and $\alpha_1 + 2\alpha_2$.]

i. Show that the vectors $\Lambda_i = \sum_{j=1}^i e_j$, $i = 1, \cdots, l-1$, $\Lambda_l = \frac{1}{2}\sum_{j=1}^l e_j$ are the fundamental weights.

[**Solution :** The fundamental weights are defined in such a way that Λ_i is orthogonal

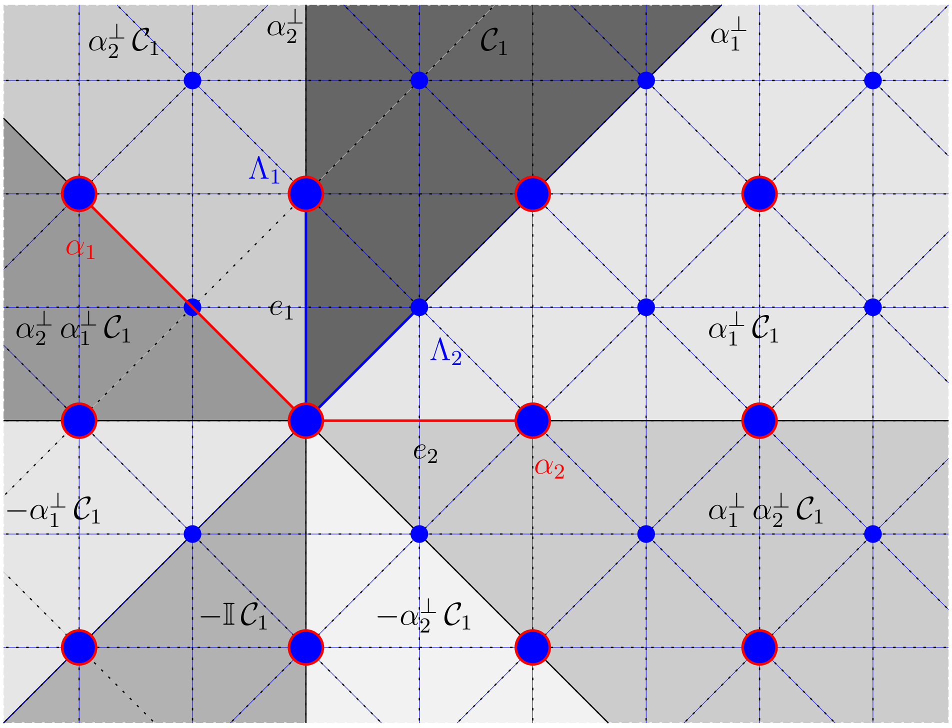

FIGURE 3 – The weight diagram for the algebra B_2. The two simple roots α_1 and α_2 are displayed as red lines. The two fundamental weights Λ_1 and Λ_2 are displayed as blue lines. The lattice of roots is shown as red dots, while the lattice of weights is shown as blue dots. The first Weyl chamber $\mathcal{C}_1$ is shown as a dark-grey zone. The other Weyl chambers, shown as grey regions of various intensities, are obtained by acting on $\mathcal{C}_1$ with the various element of the Weyl group, as explicitely indicated.

to every root simple root α_j $(j \neq i)$, its normalization being fixed by $2\frac{\langle\alpha_i,\Lambda_i\rangle}{\langle\alpha_i,\alpha_i\rangle} = 1$. The denominators read

$$\langle\alpha_i,\alpha_i\rangle = \langle e_i - e_{i+1}, e_i - e_{i+1}\rangle = 2 \quad \text{for } i = 1,\cdots,\ell-1\,,$$
$$\langle\alpha_\ell,\alpha_\ell\rangle = 1\,.$$

The construction can be easily performed by recursion. First, Λ_1 should be orthogonal to α_i $(i > 1)$, so that, from the definition of α_i, it should be orthogonal to every e_i $(i > 1)$, implying that Λ_1 is proportional to e_1. Its normalization requires that $\langle\alpha_1\,, e_1\rangle = 1$ and since $\alpha_1 = e_1 - e_2$, we thus have $\Lambda_1 = e_1$. Now, by the same reasoning, Λ_2 should be in the plane spanned by e_1 and e_2. Since Λ_2 should be orthogonal to $\alpha_1 = e_1 - e_2$, Λ_2 is proportional to $e_1 + e_2$. In the case $\ell = 2$, we thus have $\Lambda_2 = \frac{1}{2}(e_1 + e_2)$ while for $\ell > 2$, $\Lambda_2 = e_1 + e_2$. Assume now (for $i - 1 < \ell$) that From the orthogonality of Λ_i with α_j $(i < j)$ we again deduce that Λ_i should be a linear combination of e_i $(i \leq j)$. Furthermore, the orthogonality of Λ_i with α_j $(j < i)$ implies, based on the relation $e_i - e_j = \sum_{i \leq k < j} \alpha_k, \quad 1 \leq i < j \leq l$, that Λ_i is orthogonal to $e_1 - e_2, \cdots, e_1 - e_i$ and thus that Λ_i is proportional to $e_1 + e_2 + \cdots + e_i$, the coefficient of proportionality being fixed as 1 (resp. 1/2) for $i < \ell$ (resp. $i = \ell$). The weight system, and the 8 Weyl chambers, is illustrated in Fig. 3.]

j. Using Weyl formula : $\dim(\Lambda) = \prod_{\alpha>0} \frac{\langle \Lambda+\rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$ compute the dimension of the two fundamental representations of B_2 and of that of highest weight $2\Lambda_2$. In view of these dimensions, what are these representations of SO(5) ?

[**Solution** : In the case of B_2, $\Lambda_1 = e_1$, $\Lambda_2 = \frac{1}{2}(e_1 + e_2)$, $\rho = 3e_1 + e_2$. One can check that $\rho = \Lambda_1 + \Lambda_2$. The set of positive roots is $\Delta_+ = \{e_1,\ e_2,\ e_1 \pm e_2\}$. Since

$$\begin{aligned} \Lambda_1 + \rho &= \frac{5}{2} e_1 + \frac{1}{2} e_2 \,, \\ \Lambda_2 + \rho &= 2e_1 + e_2 \,, \\ 2\Lambda_2 + \rho &= \frac{5}{2} e_1 + \frac{3}{2} e_2 \,, \end{aligned}$$

we get

$$\begin{array}{rclcrcl} \langle \Lambda_1 + \rho,\, e_1 \rangle &=& \frac{5}{2}\,, & & \langle \Lambda_1 + \rho,\, e_2 \rangle &=& \frac{1}{2}\,, \\ \langle \Lambda_1 + \rho,\, e_1 - e_2 \rangle &=& \frac{5}{2} - \frac{1}{2} = 2\,, & & \langle \Lambda_1 + \rho,\, e_1 + e_2 \rangle &=& \frac{5}{2} + \frac{1}{2} = 3\,, \\ \langle \rho,\, e_1 \rangle &=& \frac{3}{2}\,, & & \langle \rho,\, e_2 \rangle &=& \frac{1}{2}\,, \\ \langle \rho,\, e_1 - e_2 \rangle &=& \frac{3}{2} - \frac{1}{2} = 1\,, & & \langle \rho,\, e_1 + e_2 \rangle &=& \frac{3}{2} + \frac{1}{2} = 2\,, \end{array}$$

and thus

$$\dim(\Lambda_1) = \prod_{\alpha>0} \frac{\langle \Lambda_1 + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \frac{\frac{5}{2} \times \frac{1}{2} \times 2 \times 3}{\frac{3}{2} \times \frac{1}{2} \times 1 \times 2}$$

i.e.

$$\boxed{\dim(\Lambda_1) = 5 \quad \text{vector representation}\,.}$$

See Fig. 4 for the corresponding weight diagram. Similarly, we deduce from

$$\begin{array}{rclcrcl} \langle \Lambda_2 + \rho,\, e_1 \rangle &=& 2\,, & & \langle \Lambda_2 + \rho,\, e_2 \rangle &=& 1\,, \\ \langle \Lambda_2 + \rho,\, e_1 - e_2 \rangle &=& 2 - 1 = 1 = 2\,, & & \langle \Lambda_2 + \rho,\, e_1 + e_2 \rangle &=& 2 + 1 = 2\,, \end{array}$$

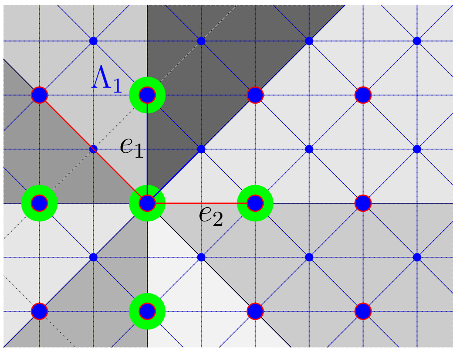

FIGURE 4 – The weight diagram for the representation (Λ_1) of the algebra B_2. The weights are shown as green dots.

that

$$\dim(\Lambda_2) = \prod_{\alpha>0} \frac{\langle \Lambda_2 + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \frac{2 \times 1 \times 1 \times 3}{\frac{3}{2} \times \frac{1}{2} \times 1 \times 2}$$

i.e.

$$\boxed{\dim(\Lambda_2) = 4 \quad \text{spinorial representation}\,.}$$

See Fig. 5 for the corresponding weight diagram. Finally,

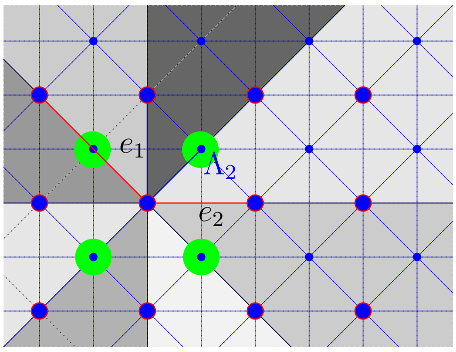

FIGURE 5 – The weight diagram for the representation (Λ_2) of the algebra B_2. The weights are shown as green dots.

$$\langle 2\Lambda_2 + \rho,\, e_1 \rangle = \frac{5}{2}\,, \qquad \langle 2\Lambda_2 + \rho,\, e_2 \rangle = \frac{3}{2}\,,$$
$$\langle 2\Lambda_2 + \rho,\, e_1 - e_2 \rangle = \frac{5}{2} - \frac{3}{2} = 1 = 2\,, \qquad \langle 2\Lambda_2 + \rho,\, e_1 + e_2 \rangle = \frac{5}{2} + \frac{3}{2} = 4\,,$$

leads to

$$\dim(2\Lambda_2) = \prod_{\alpha>0} \frac{\langle 2\Lambda_2 + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \frac{\frac{5}{2} \times \frac{3}{2} \times 1 \times 4}{\frac{3}{2} \times \frac{1}{2} \times 1 \times 2}$$

i.e.

$$\boxed{\dim(2\Lambda_2) = 10 \quad \text{adjoint representation}\,.}$$

See Fig. 6 for the corresponding weight diagram. Note that $2\Lambda_2 = \alpha_1 + 2\alpha_2$ is the

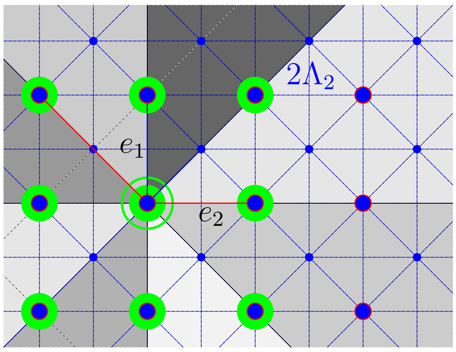

FIGURE 6 – The weight diagram for the representation $(2\Lambda_2)$ of the algebra B_2. The weights are shown as green dots.

heighest root. In the α_i basis, it reads $2\Lambda_2 = \alpha_1 + 2\alpha_2$ which sum of components is maximal. It is the heighest weight of the adjoint representation.]

k. Draw on the same figure the roots and the low lying weights of $so(5)$.

[**Solution** : See Fig. 3.]

2. $\underline{G_2}$

In the space $\mathbb{R}^2$, we consider three vectors e_1, e_2, e_3 of vanishing sum, $\langle e_i, e_j\rangle = \delta_{ij} - \frac{1}{3}$, and construct the 12 vectors

$$\pm(e_1-e_2),\ \pm(e_1-e_3),\ \pm(e_2-e_3),\ \pm(2e_1-e_2-e_3),\ \pm(2e_2-e_1-e_3),\ \pm(2e_3-e_1-e_2)$$

They make the root system of G_2, as we shall check.

a. What can be said on the dimension of the algebra G_2 ?

[**Solution** : $\dim G_2 = \text{rank} + \text{number of roots} = 12 + 2 = 14$.]

b. Show that $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_1 + e_2 + e_3$ are two simple roots, in accord with the Dynkin diagram of G_2 given in the notes. Compute the Cartan matrix.

[**Solution** : First,

$$\langle e_i, e_i\rangle = \frac{2}{3} \quad \text{and} \quad \langle e_i, e_i\rangle = -\frac{1}{3}\,,$$

which leads to

$$\langle\alpha_1,\,\alpha_1\rangle = 2\,\frac{2}{3} - 2\left(-\frac{1}{3}\right) = 2 \ \text{ and } \ \langle\alpha_2,\,\alpha_2\rangle = 4\frac{2}{3} + 2\frac{2}{3} - 4\left(-\frac{1}{3}\right) - 4\left(-\frac{1}{3}\right) + 2\left(-\frac{1}{3}\right) = 6\,,$$

and

$$\langle\alpha_1,\,\alpha_2\rangle = \langle\alpha_2,\,\alpha_1\rangle = -2\langle e_1,\,e_1\rangle + \langle e_1,\,e_2\rangle + \langle e_1,\,e_3\rangle + 2\langle e_1,\,e_2\rangle - \langle e_2,\,e_2\rangle - \langle e_2,\,e_3\rangle$$
$$= -2\frac{2}{3} - \frac{1}{3} - \frac{1}{3} - \frac{2}{3} - \frac{2}{3} + \frac{1}{3} = -3\,.$$

The Cartan matrix $C_{ij} = 2\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}$ thus reads

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

as expected, in accordance to the Dynkin diagram of Fig. 7.

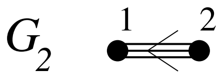

FIGURE 7 – The Dynkin diagram for the algebra G_2.

]

c. What are the positive roots ? Compute the vector ρ, half-sum of positive roots. [**Solution** : $\Delta_+ = \{\alpha_1,\ \alpha_2,\ \alpha_1{+}\alpha_2,\ 2\alpha_1{+}\alpha_2,\ 3\alpha_1{+}\alpha_2,\ 3\alpha_1{+}2\alpha_2\}$ and $\rho = 5\alpha_1{+}3\alpha_2$. The root diagram is shown in Fig. 8.]

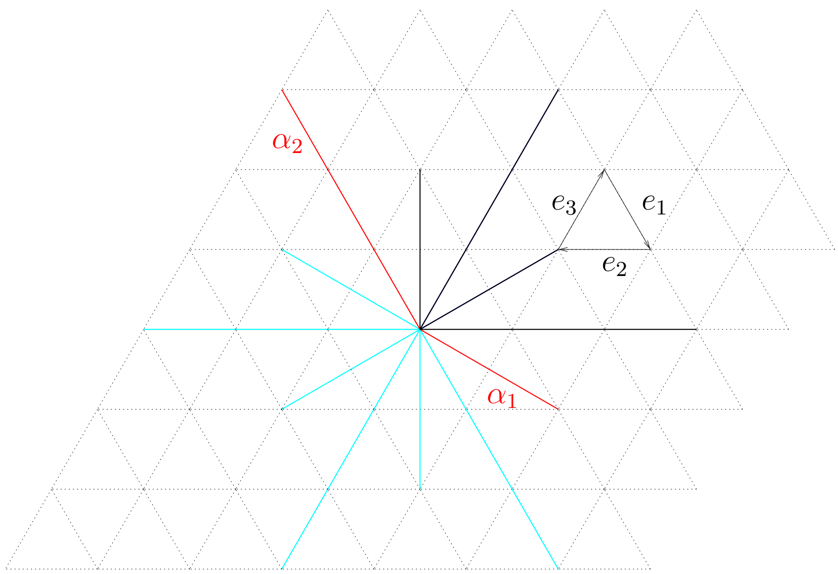

FIGURE 8 – The roots of G_2. In red, the two other simple roots α_1 and α_2. In black, the four other positive roots, and in cyan the six negative roots.

d. What is the group of invariance of the root diagram ? Show that it is of order 12 and that it is the Weyl group of G_2. Draw the first Weyl chamber.

[**Solution** : The group of invariance of the root diagram is the diedral group D_6, of order 12. The weight diagram is illustrated in Fig. 9]

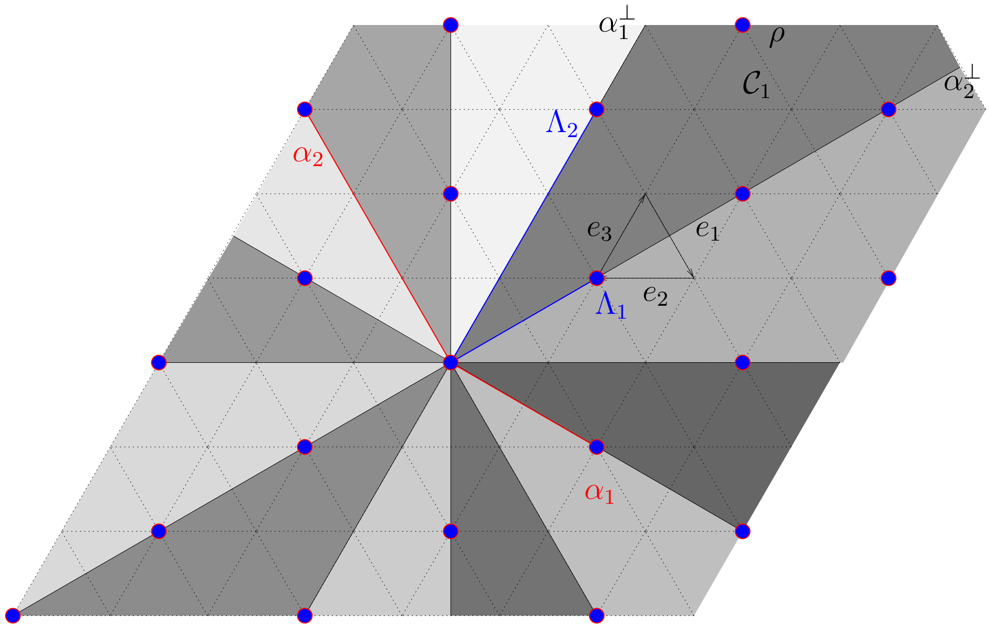

FIGURE 9 – The weight diagram for the algebra G_2. The two simple roots α_1 and α_2 are displayed as red lines. The two fundamental weights Λ_1 and Λ_2 are displayed as blue lines. The lattice of roots is shown as red dots, while the lattice of weights is shown as blue dots. These two lattices are identical for G_2. The 12 Weyl chambers (the first Weyl chambers $\mathcal{C}_1$ is explicitely indicated) are shown as grey regions of various intensities. They are obtained by acting on $\mathcal{C}_1$ with the various element of the Weyl group.

e. Check that the fundamental weights are

$$\Lambda_1 = 2\alpha_1 + \alpha_2 \qquad \Lambda_2 = 3\alpha_1 + 2\alpha_2 \,.$$

[**Solution** : First, we compute

$$\langle \alpha_1, \alpha_1 \rangle = 2\,, \quad \langle \alpha_2, \alpha_2 \rangle = 6 \quad \text{and} \quad \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = -3\,.$$

Since Λ_1 is orthogonal to α_2, obviously Λ_1 should be proportional to $2\alpha_1 + \alpha_2$. The normalization is fixed by the condition $2\frac{\langle \Lambda_1, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = 1$, leading to $\Lambda_1 = 2\alpha_1 + \alpha_2$. Similarly, Λ_2 is orthogonal to α_1, and obviously Λ_2 should be proportional to $\alpha_2 + \frac{3}{2}\alpha 1$. The normalization is fixed by the condition $2\frac{\langle \Lambda_2, \alpha_2 \rangle}{\langle \alpha_2, \alpha_2 \rangle} = 1$, leading to $\Lambda_2 = 3\alpha_1 + 2\alpha_2$.]

f. What are the dimensions of the fundamental representations?

[**Solution** :

$$\langle\rho,\,\alpha_1\rangle = 5\langle\alpha_1,\,\alpha_1\rangle + 3\langle\alpha_1,\,\alpha_2\rangle = 10-9 = 1\,,$$
$$\langle\rho,\,\alpha_2\rangle = 5\langle\alpha_1,\,\alpha_2\rangle + 3\langle\alpha_2,\,\alpha_2\rangle = -15+18 = 3\,,$$

and thus

$$\langle\rho,\,\alpha_1+\alpha_2\rangle = 4\,,\ \langle\rho,\,2\alpha_1+\alpha_2\rangle = 5\,,\ \langle\rho,\,3\alpha_1+\alpha_2\rangle = 6\,,\ \langle\rho,\,3\alpha_1+2\alpha_2\rangle = 9\,.$$

We also easily get

$$\langle\Lambda_1,\,\alpha_1\rangle = \ 2\langle\alpha_1,\,\alpha_1\rangle + \langle\alpha_1,\,\alpha_2\rangle = 4-3 = 1\,,$$
$$\langle\Lambda_1,\,\alpha_2\rangle = 2\langle\alpha_1,\,\alpha_2\rangle + \langle\alpha_2,\,\alpha_2\rangle = -6+6 = 0\,,$$

which leads to

$$\langle\Lambda_1,\,\alpha_1+\alpha_2\rangle = 1\,,\ \langle\Lambda_1,\,2\alpha_1+\alpha_2\rangle = 2\,,\ \langle\Lambda_1,\,3\alpha_1+\alpha_2\rangle = 3\,,\ \langle\Lambda_1,\,3\alpha_1+2\alpha_2\rangle = 3\,.$$

Thus

$$\dim(\Lambda_1) = \prod_{\alpha>0}\frac{\langle\Lambda_1+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} = \left(1+\frac{1}{1}\right)(1+0)\left(1+\frac{1}{4}\right)\left(1+\frac{2}{5}\right)\left(1+\frac{3}{6}\right)\left(1+\frac{3}{9}\right)$$

and thus

$$\boxed{\dim(\Lambda_1) = 7\,,}$$

see Fig. 10 for the corresponding weight diagram. Finally, we get

$$\langle\Lambda_2,\,\alpha_1\rangle = \ \ 3\langle\alpha_1,\,\alpha_1\rangle + 2\langle\alpha_1,\,\alpha_2\rangle = 6-6 = 0\,,$$
$$\langle\Lambda_2,\,\alpha_2\rangle = 3\langle\alpha_1,\,\alpha_2\rangle + 2\langle\alpha_2,\,\alpha_2\rangle = -9+12 = 3\,,$$

which leads to

$$\langle\Lambda_2,\,\alpha_1+\alpha_2\rangle = 3\,,\ \langle\Lambda_2,\,2\alpha_1+\alpha_2\rangle = 3\,,\ \langle\Lambda_2,\,3\alpha_1+\alpha_2\rangle = 3\,,\ \langle\Lambda_2,\,3\alpha_1+2\alpha_2\rangle = 6\,.$$

We obtain

$$\dim(\Lambda_2) = \prod_{\alpha>0}\frac{\langle\Lambda_2+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} = (1+0)\left(1+\frac{3}{3}\right)\left(1+\frac{3}{4}\right)\left(1+\frac{3}{5}\right)\left(1+\frac{3}{6}\right)\left(1+\frac{6}{9}\right)$$

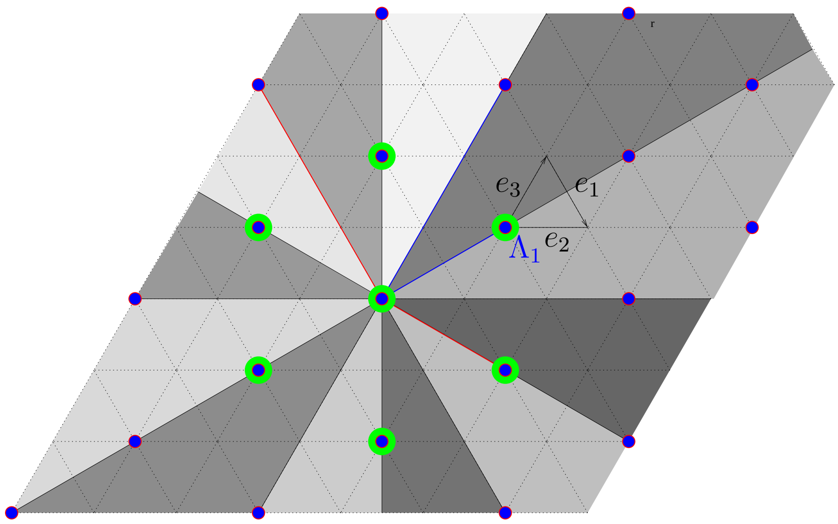

FIGURE 10 – The weight diagram for the representation (Λ_1) of the algebra G_2. The weights are shown as green dots.

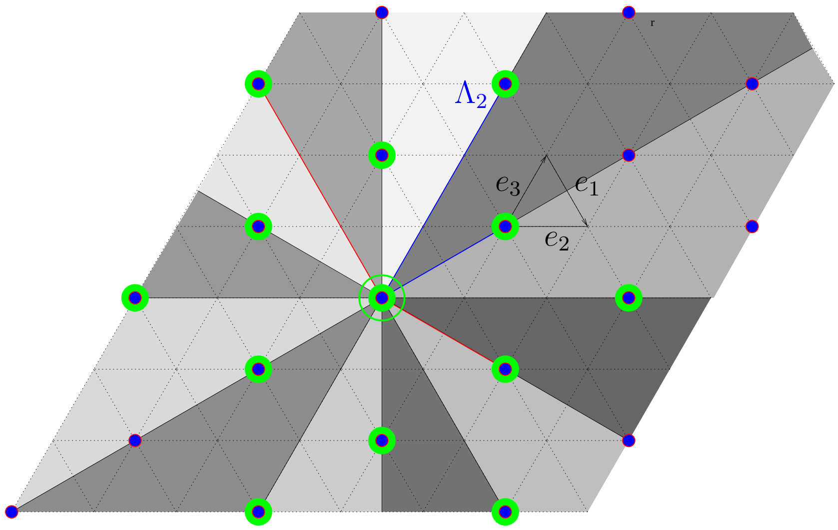

FIGURE 11 – The weight diagram for the representation (Λ_2) of the algebra G_2. The weights are shown as green dots.

and thus, as expected,

$$\boxed{\dim(\Lambda_2) = 14 \quad \text{adjoint representation}\,,}$$

see Fig. 11 for the corresponding weight diagram. Note that again, $\Lambda_2 = 3\alpha_1 + 2\alpha_2$ is the heighest root.]

g. In the two cases of B_2 and G_2, one observes that the highest weight of the adjoint representation is given by the highest root. Explain why this is true in general.

[**Solution** : The roots are the weights of the adjoint representation. The highest weight of the adjoint representation is thus the highest root.]

3. A little touch of particle physics

Why were the groups $SO(5)$ or G_2 inappropriate as symmetry groups extending the SU(2) isospin group, knowing that several "octets" of particles had been observed ?

[**Solution** : There is no representation of dimension 8, although 7+1 would be not so bad ... ?]

D. *Dimensions of SU(3) representations*

We admit that the construction of § 3.4.2, of completely symmetric traceless rank (p, m) tensors in $\mathbb{C}^3$, does give the irreducible representations of SU(3) of highest weight (p, m). Then we want to determine the dimension $d(p, m)$ of the space of these tensors.

1. Show, by studying of the product of two tensors of rank $(p, 0)$ and $(0, m)$ and separating the trace terms (those containing a δ^i_j between one lower and one upper index) that $(p, 0) \otimes (0, m) = ((p-1, 0) \otimes (0, m-1)) \oplus (p, m)$ and thus that

$$d(p, m) = d(p, 0)d(0, m) - d(p-1, 0)d(0, m-1)\,.$$

[**Solution** : The tensors of the space $(p, 0) \otimes (0, m)$ are tensors of rank (p, m) completely symmetric in their p upper indices, completely symmetric in their m lower indices, but they have a priori arbitrary traces between upper and lower indices. We would like to show that any tensor $t^{i_1\cdots i_p}_{j_1\cdots j_m}$ of this space can be expressed as the sum of a tensor with the same symmetries and with vanishing traces, and of a tensor which possesses traces, i.e. of the form

$$v^{i_1\cdots i_p}_{j_1\cdots j_m} := \sum_{n=1}^{m}\sum_{q=1}^{p} \delta^{i_q}_{j_n} u^{i_1\cdots\widehat{i_q}\cdots i_p}_{j_1\cdots\widehat{j_n}\cdots j_m}$$

where the hat on top of an index means that the given index has been omitted, and u is a tensor to be determined, which is completely symmetric in its $p-1$ upper indices and in its $m-1$ lower indices, thus belonging to the space $(p-1,0)\otimes(0,m-1)$. One would thus like to write

$$t^{i_1\cdots i_p}_{j_1\cdots j_m} = \left[t-v\right]^{i_1\cdots i_p}_{j_1\cdots j_m} + v^{i_1\cdots i_p}_{j_1\cdots j_m}\,,$$

$[t-v]$ being traceless, and we are going to fix u by requiring that $\delta^{j_1}_{i_1}[t-v]^{i_1\cdots i_p}_{j_1\cdots j_m} = 0$. (due to the symmetries of t and v this implies that every trace betwen an upper index and a lower index vanish). We first consider the case $p=m=2$, and write

$$v^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_1}\, u^{i_2}_{j_2} + \delta^{i_2}_{j_1}\, u^{i_1}_{j_2} + \delta^{i_1}_{j_2}\, u^{i_2}_{j_1} + \delta^{i_2}_{j_2}\, u^{i_1}_{j_1}\,.$$

Since each index can take 3 values, the constraint

$$\delta^{j_1}_{i_1}[t-v]^{i_1 i_2}_{j_1 j_2} = 0$$

leads to

$$t^{i i_2}_{i j_2} - [3\, u^{i_2}_{j_2} + u^{i_2}_{j_2} + u^{i_2}_{j_2} + \delta^{j_2}_{i_2}\, u^{i}_{i}] = 0\,.$$

Taking now a new trace by mean of $\delta^{j_2}_{i_2}$, we get $8\, u^i_i = t^{ij}_{ij}$. This completely fixes the expression of $u^{i_2}_{j_2}$ according to

$$5\, u^{i_2}_{j_2} = t^{i i_2}_{i j_2} - \delta^{j_2}_{i_2}\, u^i_i = t^{i i_2}_{i j_2} - \frac{1}{8}\delta^{j_2}_{i_2}\, t^{ij}_{ij}\,,$$

i.e.

$$u^{i_2}_{j_2} = \frac{1}{5} t^{i i_2}_{i j_2} - \frac{1}{40}\delta^{j_2}_{i_2}\, t^{ij}_{ij}\,.$$

One can similarly study the case $p=3$ and $m=2$, and write

$$v^{i_1 i_2 i_3}_{j_1 j_2} = \delta^{i_1}_{j_1}\, u^{i_2 i_3}_{j_2} + \delta^{i_2}_{j_1}\, u^{i_1 i_3}_{j_2} + \delta^{i_3}_{j_1}\, u^{i_1 i_2}_{j_2} + \delta^{i_1}_{j_2}\, u^{i_2 i_3}_{j_1} + \delta^{i_2}_{j_2}\, u^{i_1 i_3}_{j_1} + \delta^{i_3}_{j_2}\, u^{i_1 i_2}_{j_1}\,.$$

The contraction of $\delta^{j_1}_{i_1}$ with $[t-v]^{i_1 i_2 i_3}_{j_1 j_2}$ gives

$$[3+1+1+1] u^{i_2 i_3}_{j_2} + \delta^{i_2}_{j_2}\, u^{i i_3}_{i} + \delta^{i_3}_{j_2}\, u^{i i_2}_{i} - t^{i i_2 i_3}_{i j_2} = 0\,.$$

Next, contracting with $\delta^{i_2}_{j_2}$ gives

$$[6+3+1]\, u^{ii_3}_{i} - t^{iji_3}_{ij} = 0\,,$$

and thus allows us to fix completely u as

$$u^{i_2 i_3}_{j_2} = -\frac{1}{60}\left[t^{iji_3}_{ij}\,\delta^{i_2}_{j_2} + t^{iji_2}_{ij}\,\delta^{i_3}_{j_2}\right] + \frac{1}{6} t^{ii_2 i_3}_{ij_2}\,.$$

The general case can be treated following the same line of thought, although it is rather painful to write explicitely : after a finite number of operation (equal to $\inf(p,m)$), one can determine completely the tensor u, which achieves the proof that $(p,0)\otimes(0,m) = ((p-1,0)\otimes(0,m-1))\oplus(p,m)$ and thus that

$$d(p,m) = d(p,0)d(0,m) - d(p-1,0)d(0,m-1)\,.$$

]

2. Show by a computation analogous to that of SU(2) that

$$d(p,0) = d(0,p) = \frac{1}{2}(p+1)(p+2)\ .$$

[**Solution** : By symmetry, one can always organize the indices in such a way that 1's occurs first, then 2's and finally 3's, i.e. looking as $1\cdots1\,2\cdots2\,3\cdots3$. On should niw introduce 2 separations between the blocks of 1, 2 and 3's. Consider first the separation between 2 and 3's. There are $p+1$ possibilities (from a configuration with only 3's, looking as $3\cdots3$, till the one with no 3's, looking as $1\cdots1\,2\cdots2$), labeling for example the position r of the first 3 (which varies from 1 to $p+1$ in these two extreme cases). Then, the total number of 1's and 2's is $r-1$, and we thus have r possibilities to choose the separation between 1's and 2's. This means that the number we are looking for is just

$$d(p,0) = d(0,p) = \sum_{r=1}^{p+1} r = \frac{1}{2}(p+1)(p+2)\,.$$

]

3. Derive from it the expression of $d(p,m)$ and compare with (3.64).

[**Solution** : From $d(p,m) = d(p,0)d(m,0) - d(p-1,0)d(m-1,0) = \frac{1}{4}(p+1)(p+2)(m+1)(m+2) - \frac{1}{4}p(p+1)m(m+1)$ we deduce

$$\boxed{d(p,m) = \tfrac{1}{2}(m+1)(p+1)(m+p+2)\,,}$$

]