Moyal spaces: Metric and differential aspects

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  - Associate to any function on phase space a self-adjoint operator
  - Yields a natural construction of a deformation of the usual commutative product on algebra of functions on phase space:
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  - Called $\star$-product or Moyal product (dev. by Moyal $\sim$ 1949)
- Physics: Decreasing interest $>$ 1960 - Mathematical developments in deformation theory

Moyal spaces - Motivations - II

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- Moyal-product: integral formula among functions of $\mathcal{S}(\mathbb{R}^n)$ with $\mathbb{R}^n$ translation
- Rieffel deformation theory: $\star$-product particular case of "generalised" $\star$-product from isometric action α of some $\mathbb{R}^p$ on C* alg. of functions (plus given $\Theta \in \mathbb{M}_p(\mathbb{C})$), $\star_\Theta$.
  - i) M compact Riemann and α periodic: Periodic isospectral deformation, $((C(M),.),L^2(M,S),D)$ —> $((C(M),\star_\Theta),L^2(M,S),D)$
  - ii) Generalisable to non compact M - $\alpha_{\mathbb{R}^n}$: (non compact) non periodic isospectral deformation

Moyal spaces - Motivations - III

- Moyal plane can be viewed as non periodic isospectral deformation of $\mathbb{R}^2$ $((\mathcal{S}(\mathbb{R}^2),\star), L^2(\mathbb{R}^2)\otimes\mathbb{C}^2, D)$ where D is usual Dirac operator on $\mathbb{R}^2$.

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- 2000: Evidence for new problem with renormalisability of field theories on noncommutative Moyal spaces.
- Interesting in mathematical physics: pathologies expected to be generic of many type of noncommutative field theories.

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- 2007: Construction of candidate for renormalisable NC gauge theory in $d = 4$ [de Goursac, Wallet, Wulkenhaar]
- 2008: Vacuum configurations for the above theory in $d = 2$ and $d = 4$ [de Goursac, Wallet]

Some features of Moyal spaces

- Part 1: Metric aspects of noncommutative Moyal geometry
  - i) Can one characterize properties of the spectral distance on Moyal planes?
  - ii) Can one possibly compute explicit distance formula?

  First exemple of explicit spectral distance formula on a not almost commutative space (or finite space)
  Extend to other triples (Poddleś, Tori, $SU_q(2)$) (under study)
  D'Andrea, Martinetti, Wallet

- Part 2: NC gauge theories on a nutshell.

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   - Definition
   - Exemples
   - Other exemples
2. **MOYAL - BASICS**
   - The Moyal product
   - The matrix base
   - Usefull properties of the matrix base
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   - Theorem
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   - Truncation of the Moyal Triple
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   - Derivation-based differential calculus

THE SPECTRAL DISTANCE

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- Definition
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2 MOYAL - BASICS

3 DISTANCE ON MOYAL PLANE

4 DISCUSSION

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6 Noncommutative Torus - preliminaries

Spectral triple and Spectral distance

Definition 1

Spectral triple is $(\mathbb{A}, \mathcal{H}, D)$ with:
i) $\mathbb{A}$, associative involutive algebra, represented faithfully $\pi : \mathbb{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ (separable) Hilbert
ii) D selfadjoint not necessarely bounded, defined on $\text{Dom}(D)$ dense in $\mathcal{H}$
iii) For any $a \in \mathbb{A}$, $\pi(a)(D-\lambda)^{-1} \in \mathcal{K}(\mathcal{H})$, $\forall \lambda \notin \text{Sp}(D)$
iv) For any $a \in \mathbb{A}$, $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$

- Supplemented by additional conditions. Will be discused below for non compact Moyal triple [Gayral,Gracia-Bondia,Iochum,Schücker,Varilly 2004]
- What is needed to actually compute the distance is $(\mathbb{A}, \mathcal{H}, D)$.

Definition 2 (Connes, 1994)

A spectral triple $(\mathbb{A}, \mathcal{H}, D)$ induces a distance on the space of states $\mathcal{S}(\mathbb{A})$ defined by

$$d(\omega_1, \omega_2) = \sup_{a \in \mathbb{A}} \left(|\omega_1(a) - \omega_2(a)|, \ ||[D, \pi(a)]|| \leq 1 \right) \tag{1}$$

for any $\omega_1, \omega_2 \in \mathbb{A}$.

Exemples

- Commutative triple coding M compact Riemann spin:
  d coincides with Riemann distance on M. Initial observation (Connes)
  - 2 ways to get Riemann distance:($\gamma : [0,1] \to M$, $\gamma(0) = \omega_1$, $\gamma(1) = \omega_2$)

$$d_g(\omega_1, \omega_2) := \inf_{\gamma} \big(l(\gamma) \big) = \sup_{a \in C(M)} \big(|a(\omega_1) - a(\omega_2)|, \ ||[D,a]|| \leq 1 \big)$$

LHS: Trajectories; RHS: operatorial. $||[D,a]|| = ||\nabla a||_\infty$, $a \in C(M)$.
Commutative case: points are pure states ($a(\omega) = \omega(a)$).

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- $(\mathbb{A} = \mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, D)$. Pure states are end points of $[0,1]$.
  - $d = \frac{1}{|a|}$ for D =antidiag(a, a) ($a \neq 0$) - $d = +\infty$ for D =diag(a, b).

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- $(\mathbb{M}_2(\mathbb{C}), \mathbb{C}^2, D)$, sp$D = (\alpha, \beta), \alpha \neq \beta$ [Krajewski,Iochum,Martinetti 2001] - Pure states: $CP^1 \sim \mathbb{S}^2$

$$d(P, Q) = \frac{1}{|\alpha - \beta|}((x_P - x_Q)^2 + (y_P - y_Q)^2)^{1/2}, \ z_P = z_Q$$

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- First rudimentary exemple from physicists: 1-D lattice (Dirac operator=finite difference operator) d ∼lattice spacing [Dimakis, Müller-Hoissen 1993; Bimonte, Lizzi, 1994]

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- Proposed criterium for Compact Quantum Metric Space [Rieffel 1998-2003].

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- Proposed criterium for Compact Quantum Metric Space [Rieffel 1998-2003].
- The present contruction: First explicit distance formula on non trivial noncommutative space: Moyal space.

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The Moyal product

- $\mathcal{S}(\mathbb{R}^2) \equiv \mathcal{S}$: Schwarz functions, $\mathcal{S}'(\mathbb{R}^2) \equiv \mathcal{S}'$, $||.||_2$, $\langle .,. \rangle$: $L^2(\mathbb{R}^2)$ norm and inner product.

Definition 3

Associative bilinear Moyal $\star$-product defined as: $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $\forall a, b \in \mathcal{S}$

$$(a \star b)(x) = \frac{1}{(\pi\theta)^2} \int d^2y d^2t \ a(x+y)b(x+t)e^{-i2y\Theta^{-1}t}$$

$$y\Theta^{-1}t \equiv y^\mu \Theta^{-1}_{\mu\nu} t^\nu, \ \Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \theta \in \mathbb{R}, \ \theta \neq 0$$

Proposition 4 (see e.g Gracia-Bondia, Varilly, 1988)

One has:
i) $(a \star b)^\dagger = b^\dagger \star a^\dagger$
ii) $(a, b) := \int d^2x \ (a \star b)(x) = \int d^2x \ (b \star a)(x) = \int d^2x \ a(x)b(x)$
iii) $\partial_\mu (a \star b) = \partial_\mu a \star b + a \star \partial_\mu b$.
iv) $\mathcal{A} \equiv (\mathcal{S}, \star)$ *is a non unital associative involutive Fréchet algebra.*

-

The matrix base

- Natural basis for $(\mathcal{S},\star)$:

Definition 5

Matrix base: family of functions $\{f_{mn}\}_{m,n\in\mathbb{N}}\subset\mathcal{S}$ such that

$$H\star f_{mn}=\theta(m+\frac{1}{2})f_{mn},\ f_{mn}\star H=\theta(n+\frac{1}{2})f_{mn},\ H=\frac{1}{2}(x_1^2+x_2^2),\ \forall m,n\in\mathbb{N}$$

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- Usefull properties (Set $\bar{z}=\frac{1}{\sqrt{2}}(x_1-ix_2)$, $z=\frac{1}{\sqrt{2}}(x_1+ix_2)$.)

Proposition 6

$\{f_{mn}\}_{m,n\in\mathbb{N}}$ with $f_{mn}=\frac{1}{(\theta^{m+n}m!n!)^{1/2}}\bar{z}^{\star m}\star f_{00}\star z^{\star n}$, $f_{00}=2e^{-2H/\theta}$. One has :

$$f_{mn}\star f_{pq}=\delta_{np}f_{mq},\ \ f_{mn}^*=f_{nm},\ \ \langle f_{mn},f_{kl}\rangle=(2\pi\theta)\delta_{mk}\delta_{nl} \tag{2}$$

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$$f_{mn}\star f_{pq} = \delta_{np}f_{mq},\ \ f^*_{mn} = f_{nm},\ \ \langle f_{mn}, f_{kl}\rangle = (2\pi\theta)\delta_{mk}\delta_{nl} \tag{2}$$

- Usefull isomorphism

Proposition 7 (Gracia-Bondia, Varilly, 1988)

Frechet algebra isomorphism between $\mathcal{A}\equiv(\mathcal{S},\star)$ *and matrix algebra of decreasing sequences* (a_{mn}), $\forall m,n\in\mathbb{N}$ *defined by* $a = \sum_{m,n} a_{mn}f_{mn}$, $\forall a\in\mathcal{S}$, *such that the semi-norms* $\rho_k^2(a)\equiv\sum_{m,n}\theta^{2k}(m+\frac{1}{2})^k(n+\frac{1}{2})^k|a_{mn}|^2<\infty$, $\forall k\in\mathbb{N}$.

The matrix base - II

- Within matrix base, star product is like "matrix product". For $a=\sum_{m,n}a_{mn}f_{mn}$, $b=\sum_{m,n}b_{mn}f_{mn}$, $a,b\in\mathcal{S}$, sequences $c_{mn}=\sum_p a_{mp}b_{pn}$, $\forall m,n\in\mathbb{N}$ define the function $c=\sum_{m,n}c_{mn}f_{mn}:=a\star b$.

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- Usefull property (Set $L_a(b) := a \star b$)

Proposition 8

For any $a,b \in L^2(\mathbb{R}^2)$, $a \star b \in L^2(\mathbb{R}^2)$, $||a \star b||_2 \le \frac{1}{2\pi\theta}||a||_2||b||_2$ so that $||L_a|| \le \frac{1}{2\pi\theta}||a||_2$.

Proof.

Use matrix base and Cauchy-Schwartz inequality. □

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Proposition 9 (Gracia-Bondia, Varilly, 1988)

$(\mathcal{A},\star)$ is a pre-C algebra.*

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The Moyal spectral triple

Set $\partial := \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2)$, $\bar{\partial} := \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2)$.

Proposition 10

$(\mathcal{A} := (\mathcal{S}, \star)$, $\mathcal{H} := L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$, $D := -i\partial_\mu \otimes \sigma^\mu)$ with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ D = -i\sqrt{2}\begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix}$$

is a spectral triple.

- D usual Dirac operator on $\mathbb{R}^2$. Self-adjoint, densely defined on $\text{Dom}(D) = (\mathcal{D}_{L^2} \otimes \mathbb{C}^2)$.
- Left regular representation: $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\pi(a) := L_a \otimes \mathbb{I}_2$. $\pi(a)\psi = (a \star \psi_1, a \star \psi_2)$, $\forall \psi = (\psi_1, \psi_2) \in \mathcal{H}$, $\forall a \in \mathcal{A}$.
- For any $a \in \mathcal{S}$, $\pi(a) \in \mathcal{B}(\mathcal{H})$ (Prop. 8). $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$ in view of
$$[D, \pi(a)]\psi = -i\sqrt{2}\begin{pmatrix} L_{\partial a} & 0 \\ 0 & L_{\bar{\partial} a} \end{pmatrix}\begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}, \ \forall \psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H}$$
- $\pi(a)(D - \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$, $\forall a \in \mathcal{A}$, $\forall \lambda \notin \text{Sp}(D)$. Use:$\pi(a)(D - i\rho)^{-1} \in \mathcal{K}(\mathcal{H}) \iff \pi(a)(D^2 + \rho^2)^{-1} \in \mathcal{K}(\mathcal{H})$ ($\rho \in \mathbb{R}^*$ and observe that for any $a, b \in L^2(\mathbb{R}^2)$, $L(a)b(-i\nabla)$ is Hilbert-Schmidt [B.Simons].

The Moyal spectral triple - II

Additional algebraic conditions:
- Self-adjoint operator on $\mathcal{H}$ $\chi := \mathbb{I} \otimes (-i\sigma^1\sigma^2)$. $\chi^2 = 1$, defines a $\mathbb{Z}_2$-grading of $\mathcal{H}$ and $D\chi = -\chi D$. (Physics: Like chirality operator)
- $J : \mathcal{H} \to \mathcal{H}$, $J := \mathbb{I} \otimes (-i\chi\sigma^1).\kappa$ (Physics: Like charge conjugation operation). $J^2 = -1$, $DJ = JD$, $J\chi = -\chi J$. One checks: $[a, Jb^* J^{-1}] = 0$ and $[[D, \pi(a)], Jb^* J^{-1}] = 0$ (1st order condt.), $\forall a, b \in \mathcal{A}$.
- Spectral dimension $=2$. [Gayral, Gracia-Bondia, Iochum, Schücker,Varilly, 2004]

Proposition 11

$((\mathcal{A}, \mathcal{H}, D); \chi, J)$ *is an even real spectral triple with spectral dimension 2.*

Compute spectral distance formula between two pure states.

Pure states

- Very convenient to use the matrix base.
  - Observe: Representation of $\mathcal{A}$ in the triple reducible. $\mathcal{G}_N :=$Span$(f_{mN})_{m\in\mathbb{N}}$, N fixed, invariant under left action of $\mathcal{A}$.
  - Vector state: $\omega_{mn}(a) := \frac{1}{2\pi\theta} < f_{mn}, L_a f_{mn} >= a_{mm}$. Depends only on "first index". Then, fix $N = 0$. Work with $\mathcal{G}_0$.

Proposition 12

The pure states of $\bar{\mathcal{A}}$ are the vector states $\omega_\psi : \bar{\mathcal{A}} \to \mathbb{C}$ defined by any unit vector $\psi \in L^2(\mathbb{R}^2)$ of the form $\psi = \sum_{m\in\mathbb{N}} \psi_m f_{m0}$, $\sum_{m\in\mathbb{N}} |\psi_m|^2 = \frac{1}{2\pi\theta}$ and one has

$$\omega_\psi(a) \equiv \big\langle(\psi, 0), \pi(a)(\psi, 0)\big\rangle = 2\pi\theta \sum_{m,n\in\mathbb{N}} \psi_m^* \psi_n a_{mn} \tag{3}$$

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$$\omega_\psi(a) \equiv \langle(\psi,0),\pi(a)(\psi,0)\rangle = 2\pi\theta \sum_{m,n\in\mathbb{N}} \psi_m^*\psi_n a_{mn} \tag{3}$$

Proof.

i) Show that $\bar{\mathcal{A}}$ is $\star$-isomorphic to $\mathcal{K}(\mathcal{G}_0)$.
ii) The result follows from Lemma [Kadison, II, p.750]: Let $\mathbb{A}$, a sub-C* of $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H}) \subseteq \mathbb{A}$, ρ a pure state of $\mathbb{A}$. Then, either $\rho = 0$ or ρ is vector state generated by some unit vector in $\mathcal{H}$. □

Usefull lemma

- Define: $\mathcal{B}_1 := \{a \in \mathcal{A},\ /\ ||[D,\pi(a)]|| \leq 1\}$.
- Lemma extends to NC torus, Podleś sphere, $SU_q(2)$ [CW 2009]

Lemma 13 (CW 2009, CDMW 2009)

We set $\partial a = \sum_{m,n} \alpha_{mn} f_{mn}$ *and* $\bar{\partial} a = \sum_{m,n} \beta_{mn} f_{mn}$, *for any* $a \in \mathcal{A}$.
Assume that $a \in \mathcal{B}_1$. *Then:*
i) $|\alpha_{mn}| \leq \frac{1}{\sqrt{2}}$ *and* $|\beta_{mn}| \leq \frac{1}{\sqrt{2}}$, $\forall m,n \in \mathbb{N}$.
ii) Define $\hat{a}(m_0) := \sum_{p,q\in\mathbb{N}} \hat{a}_{pq}(m_0) f_{pq}$ *with*

$$\hat{a}_{pq}(m_0) = \delta_{pq}\sqrt{\frac{\theta}{2}}\sum_{k=p}^{m_0}\frac{1}{\sqrt{k+1}}, \text{ with fixed } m_0 \in \mathbb{N}.$$

Let $\mathcal{A}_+$ *denotes the set of positive elements of* $\mathcal{A}$. *Then,* $\hat{a}(m_0) \in \mathcal{A}_+$ *and* $||[D,\pi(\hat{a}(m_0))]||_{op} = 1$ *for any* $m_0 \in \mathbb{N}$.

Proof.

Go to

A spectral distance formula on the Moyal plane

Definition 14

We denote by ω_m the pure state generated by the unit vector $\frac{1}{\sqrt{2\pi\theta}} f_{m0}$, $\forall m \in \mathbb{N}$. For any $a = \sum_{m,n} a_{mn} f_{mn} \in \mathcal{A}$, one has $\omega_m(a) = a_{mm}$.

▶

Theorem 15 (CDMW 2009)

The spectral distance between any two pure states ω_m and ω_n is

$$d(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{k=n+1}^{m} \frac{1}{\sqrt{k}}, \ \forall m, n \in \mathbb{N}, \ n < m. \qquad (4)$$

It verifies the "triangular equality"

$$d(\omega_m, \omega_n) = d(\omega_m, \omega_p) + d(\omega_p, \omega_n), \ \forall m, n, p \in \mathbb{N}. \qquad (5)$$

▶

Proof of theTheorem

Proof.

Algebraic property of matrix base yields

$$\alpha_{n+1,n} = \sqrt{\frac{n+1}{\theta}}(a_{n+1,n+1} - a_{n,n}) = \sqrt{\frac{n+1}{\theta}}(\omega_{n+1}(a) - \omega_n(a)),\ \forall n \in \mathbb{N}.$$

Use Lemma: for any a in the unit ball, $|\omega_{n+1}(a) - \omega_n(a))| \leq \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$ so $d(\omega_{n+1}, \omega_n) \leq \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$. This bound is saturated by any $\hat{a}(m_0)$, $m_0 \geq n$, $m_0, n \in \mathbb{N}$ defined in Lemma.Therefore: $d(\omega_{n+1}, \omega_n) = \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$. Now, triangular inequality: $d(\omega_m, \omega_n) \leq \sum_{k=n}^{m-1} d(\omega_k, \omega_{k+1})$, (assuming $n < m$). Upper bound saturated by any $\hat{a}(m_0)$, $m_0 \geq n$. Consider $|\omega_m(\hat{a}(m_0)) - \omega_n(\hat{a}(m_0)))|$, $n < m \leq m_0$.

$$|\omega_m(\hat{a}(m_0)) - \omega_n(\hat{a}(m_0)))| = \sqrt{\frac{\theta}{2}}|\sum_{k=m}^{m_0}\frac{1}{\sqrt{k+1}} - \sum_{k=n}^{m_0}\frac{1}{\sqrt{k+1}}| = \sqrt{\frac{\theta}{2}}\sum_{k=n}^{m-1}\frac{1}{\sqrt{k+1}} \quad (6)$$

Therefore, $d(\omega_m, \omega_n)$ satisfies (5). Relation (4) follows immediately. □

DISCUSSION

1. THE SPECTRAL DISTANCE
2. MOYAL - BASICS
3. DISTANCE ON MOYAL PLANE
4. DISCUSSION
   - States at infinite distance
   - Consequences
   - Truncation of the Moyal Triple
   - A spectral distance on the 2-sphere
5. GAUGE THEORIES ON MOYAL SPACES: RESULTS
6. Noncommutative Torus - preliminaries

Some states at finite distance

- Some states at finite distance to each other

Proposition 16

Let $\mathcal{I}$ be a finite subset of $\mathbb{N}$ and let $\Lambda = \sum_{m\in\mathcal{I}\subset\mathbb{N}} \lambda_m f_{m0}$ denotes a unit vector of $L^2(\mathbb{R}^2)$. Then $d(\omega_n, \omega_\Lambda) < +\infty$, for any $n \in \mathbb{N}$.

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Proof.

For any $n\in\mathbb{N}$, and any $a\in\mathcal{A}$, including any element of the unit ball, one has

$$|\omega_\Lambda(a)-\omega_n(a)|=|2\pi\theta\sum_{p,q\in\mathcal{I}}a_{pq}\lambda^\star_p\lambda_q-a_{nn}|\le 2\pi\theta\sum_{p,q\in\mathcal{I}}|a_{pq}||\lambda^\star_p\lambda_q|+|a_{nn}|$$
$$\le\sum_{p,q\in\mathcal{I}}|a_{pq}|+|a_{nn}|$$

(last inequality from: $|\lambda_n|\le\frac{1}{\sqrt{2\pi\theta}}$, $\forall n\in\mathcal{I}$). Simple algebraic property of matrix base: a_{mn}'s expressible as finite sums of α_{mn} and β_{mn}. Unit ball: $|\alpha_{mn}|\le\frac{1}{\sqrt{2\pi\theta}}$ and $|\beta_{mn}|\le\frac{1}{\sqrt{2\pi\theta}}$. Therefore RHS bounded. □

States at infinite distance

- There are pure states at infinite distance

Definition 17

Let $\psi(s)$ family of unit vectors of $L^2(\mathbb{R}^2)$ defined by
$\psi(s) := \frac{1}{\sqrt{2\pi\theta}} \sum_{m\in\mathbb{N}} \sqrt{\frac{1}{\zeta(s)(m+1)^s}} f_{m0}$ for any $s \in \mathbb{R}$, $s > 1$ ($\zeta(s)$ Riemann zeta function). Corresponding family of pure states denoted by $\omega_{\psi(s)}$, for any $s \in \mathbb{R}$, $s > 1$, with $\omega_{\psi(s)}$ as in Proposition 12.

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- At infinite distance from any ω_m

Proposition 18 (CDMW 2009)

$d(\omega_n, \omega_{\psi(s)}) = +\infty$, $\forall s \in]1, \frac{3}{2}]$, $\forall n \in \mathbb{N}$.

Proof of Proposition 18

Proof.

$$B(m_0;\psi,\psi') := |\omega_{\psi'}(\hat{a}(m_0)) - \omega_\psi(\hat{a}(m_0))| \le d(\omega_\psi, \omega'_\psi),\ \forall m_0 \in \mathbb{N}. \qquad (7)$$

First pick $\psi = \frac{1}{\sqrt{2\pi\theta}} f_{00} := \psi_0$. Assume that $\psi' = \psi(s)$. Then:

$$B(m_0;\psi_0,\psi(s)) = \sqrt{\frac{\theta}{2}} \Big| \sum_{m=0}^{m_0} \sum_{k=m}^{m_0} \frac{1}{\sqrt{k+1}} \frac{1}{\zeta(s)(m+1)^s} - \sum_{k=0}^{m_0} \frac{1}{\sqrt{k+1}} \Big|. \qquad (8)$$

Next: "$\sum_{k=m}^{m_0} = \sum_{k=0}^{m_0} - \sum_{k=0}^{m}$"

$$B(m_0;\psi_0,\psi(s)) = \sqrt{\frac{\theta}{2}} \Big| A_1(m_0) + \frac{1}{\zeta(s)} \sum_{m=0}^{m_0} \sum_{k=0}^{m} \frac{1}{(m+1)^s \sqrt{k+1}} \Big|. \qquad (9)$$

$A_1(m_0)$ positive term. Then observe

$$\frac{1}{\zeta(s)} \sum_{m=0}^{m_0} \sum_{k=1}^{m+1} \frac{1}{(m+1)^s \sqrt{k}} \ge \frac{1}{\zeta(s)} \sum_{m=0}^{m_0} \frac{\sqrt{m+1}}{(m+1)^s}, \qquad (10)$$

use $\sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} \ge 2(\sqrt{m+2} - 1)$. $A_2(m_0)$ bounded below by quantity divergent when m_0 goes to $+\infty$ whenever $s \le \frac{3}{2}$. Therefore: $d(\omega_0, \omega_{\psi(s)}) = +\infty$, $\forall s \in]1, \frac{3}{2}]$. Triangular inequality $d(\omega_0, \omega_{\psi(s)}) \le d(\omega_0, \omega_n) + d(\omega_n, \omega_{\psi(s)})$, for any $n \in \mathbb{N}$ terminates the proof. □

States at infinite distance

- Distance among the $\omega_{\psi(s)}$'s is infinite.

Proposition 19 (CW 2010)

$d(\omega_{\psi(s_1)},\omega_{\psi(s_2)})=+\infty,\ \forall s_1,s_2\in]1,\frac{5}{4}[\cup]\frac{5}{4},\frac{3}{2}],\ s_1\neq s_2.$

Proof.

Repeated use of mean value theorem to obtain estimates of $|\omega_{\psi'}(\hat{a}(m_0))-\omega_{\psi}(\hat{a}(m_0))|$. □

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Proof.

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Proposition 20

For any state, there is at least another state which is at infinite distance.

Consequences

- Topology induced by the spectral distance d on space of states of $\bar{\mathcal{A}}$ not the weak * topology.

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- Notice: Unitalization of $(\mathcal{A}, \mathcal{H}, D)$ in [Gayral et al., 2004] uses $\mathcal{A}_1$: algebra of bounded functions with all bounded derivatives. $(\mathcal{A}_1, \mathcal{H}, D)$ is not CQMS, despite $\mathcal{A}_1$ unital.

Consequences

- Topology induced by the spectral distance d on space of states of $\bar{A}$ not the weak * topology.
- Weak* Topology : Basic condition to have Compact Quantum Metric Spaces (CQMS) as defined by Rieffel
- Spectral Triple proposed by [Gayral et al., 2004] built from $(\mathcal{A}, \mathcal{H}, D)$ is of course not CQMS (proposed as NC analog of non compact Riemann spin geometry).
- Notice: Unitalization of $(\mathcal{A}, \mathcal{H}, D)$ in [Gayral et al., 2004] uses $\mathcal{A}_1$: algebra of bounded functions with all bounded derivatives. $(\mathcal{A}_1, \mathcal{H}, D)$ is not CQMS, despite $\mathcal{A}_1$ unital.
- Could one "truncate" the spectral triple to get a CQMS? Not sufficient to have finite dimensional algebra: Spectral triple of [Iochum,Krajewski,Martinetti, 2001] built from $\mathbb{M}_2(\mathbb{C})$ with different D is not CQMS. Recall corresponding spectral distance on $\mathbb{S}^2$ is infinite for pure states (points) at different "altitude".

Truncation of the Moyal Triple

Set $\mathcal{A}_N := \mathbb{M}_N(\mathbb{C})$. Inner product $< a, b >_\theta = 2\pi\theta \mathrm{Tr}(a^\dagger b)$.

Proposition 21

The following data define a spectral triple $(\mathcal{A}_N, \mathcal{H}_N = \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}^2, D_N)$, $\partial a = -[X_-, a]$, $\bar{\partial} a = [X_+, a]$

$$X_- := \frac{1}{\sqrt{\theta}} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \sqrt{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \sqrt{n-1} & 0 \end{pmatrix}, \qquad X_+ := X_-^t. \tag{11}$$

For $N \to +\infty$, one recovers D.
Re-express the spectral distance.

$$d(\omega_A, \omega_B) = \sup_{a \in \mathcal{V}_N} \left\{ |\omega_A(a) - \omega_B(a)|,\ ||\partial a||_{\leq} \frac{1}{\sqrt{2}} \right\} := d(A, B) \tag{12}$$

$A_1, A_2 \in \mathcal{S}(\mathcal{A}_N) = \{A \in \mathcal{A}_N^+,\ Tr(A) = 1\}$, $\mathcal{V}_N = \{a \text{ self-adjoint, traceless}\}$, $\omega_A(a) = Tr(Aa)$

Equivalent distances

Define $\Delta(A,B) := ||A-B||_{L^2}$, $||A||^2_{L^2} :=< A, A >$

Proposition 22

$d(A,B)$ and $\Delta(A,B)$ are equivalent.

Proof.

Observe: i) $\Delta(A,B) = \sup_{a\in\mathcal{V}_N}(|\omega_A(a)-\omega_B(a)|,\ ||a||_{L^2}\leq 1)$; ii) $||.||_{op}$ and $||||_{L^2}$ are equivalent norms on $\mathcal{V}_N$ □

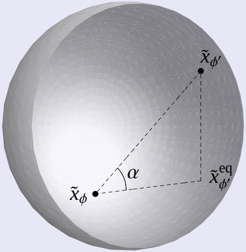
Lemma 23 (CDMW 2009)

$(\mathcal{A}_N, \mathcal{H}_N = \mathbb{M}_N(\mathbb{C})\otimes\mathbb{C}^2, D_N)$ defines a CQMS.

Proof.

$\Delta(A,B)$ induces weak* topology on $\mathcal{S}(\mathcal{A}_N)$. □

Exemple: Spectral distance on the 2-sphere



Proposition 24

$$d(\omega_\phi, \omega_{\phi'}) = \cos\alpha\, d_{eucl}(x_\phi, x_{\phi'}) \text{ for } \alpha \leq \frac{\pi}{4}$$
$$d(\omega_\phi, \omega_{\phi'}) = \frac{1}{2\sin\alpha} d_{eucl}(x_\phi, x_{\phi'}) \text{ for } \alpha \geq \frac{\pi}{4}$$

GAUGE THEORIES ON MOYAL SPACES: RESULTS

1. THE SPECTRAL DISTANCE
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Derivation-based differential calculus and Moyal gauge theory

- $\mathbb{A} = \mathcal{M}$, Left-$\mathbb{A}$ module $\mathbb{M} = \mathcal{M}$. Der($\mathcal{M}$) =Int($\mathcal{M}$). One has $\partial_\mu = i[\xi_\mu,]_\star$, $\xi_\mu = -\Theta^{-1}_{\mu\nu} x^\nu$. General framework: derivation based differential calculus [Connes 1980, Dubois-Violette 1986]: $\{\partial_\mu\}$. Let $\hat{d}$ the differential.
- Connection defined by $\nabla : \Omega^0 \to \Omega^1$, $\nabla a = \hat{d}a + Aa$, $A \in \Omega^1$. Hermitian connection: $Xh(a_1, a_2) = h(\nabla_X(a_1), a_2) + h(a_1, \nabla_X(a_2))$, $\forall a_1, a_2 \in \mathcal{M}$, $\forall X \in$Der($\mathcal{M}$), with $h_0(a_1, a_2) = a_1^\dagger a_2$. Set $\nabla_X(\mathbb{I}) := -iA_\mu$. Curvature $F(a) = (\hat{d}A + AA)a$, $F(X, Y)a = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})a$. Unitary gauge group $\mathcal{U}(\mathcal{M}) \subset$Aut($\mathcal{M}$), $h(a_1^g, a_2^g) = h(a_1, a_2)$.
- **A simple Lemma** [e.g Cagnache,Masson, Wallet 2008]: Assume $\exists \eta \in \Omega^1$ $\hat{d}a = [\eta, a]$, $\forall a \in \Omega^\bullet$. Then $\nabla^{inv} := \hat{d}a - \eta a$ defines a connection. It is invariant under unitary gauge transformations. The "covariant coordinates" are nothing but the tensor 1-form defined by $\mathcal{A} := \nabla - \nabla^{inv} = A + \eta$.
- 4-d Candidate action for renormalisable gauge theory on Moyal space [de Goursac,Wallet,Wulkenhaar 2007]: Curvature is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star = i[\mathcal{A}_\mu, \mathcal{A}_n u] - \Theta^{-1}_{\mu\nu}$

$$S = \int d^4x(\alpha F_{\mu\nu} \star F_{\mu\nu} + \omega\{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \kappa \mathcal{A}_\mu \star \mathcal{A}_\mu)$$

Gauge theory on Moyal space

- Vacuum configuration (2-d and 4-d) highly non trivial when action is **not regarded** as a matrix model (Situation prefered by physicists!).

Proposition 25 (de Goursac, Wallet 2008)

The vacuum configuration in 4 dimensional case is

$$\mathcal{A}^0_\mu(x) = 2\sqrt{2}\theta\frac{e^{z/2}}{z}\int_{t=0}^{\infty} e^{-t}J_2(2\sqrt{tz})\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sqrt{v_{m+1}}\,t^{m+1}(\tilde{x}_\mu\cos(\xi_m)+\frac{2}{\theta}\sin(\xi_m)) \quad (13)$$

where v_m and ξ_m defined in EPJC 2008 and $z=\frac{2x^2}{\theta}$.

- Derivation based differential calculus can be extended to $\mathbb{Z}_2$-graded case (Wallet 2008). Yields interesting exemple of gauge theory constructed from $(\mathbb{A}=\mathcal{M}\oplus\mathcal{M},\bullet)$ and differential calculus based on derivations generated by polynomials of $d^0\leq 2$ of one $\mathcal{M}$ suitably completed to make a Lie algebra of derivations of $\mathbb{A}$.

Proposition 26 (Wallet 2008)

The renormalisable 4-d φ^4 theory is a truncation of a gauge theory built from the above differential calculus.

Remark

- Equivalence classes:

Definition 27

For any states ω_1 and ω_2, denote by $\approx$ the equivalence relation $\omega_1 \approx \omega_2 \iff d(\omega_1, \omega_2) < +\infty$. $[\omega]$ denotes the equivalence class of ω.

- Several equivalence classes: $[\omega_n] = [\omega_0]$, $\forall n \in \mathbb{N}$. $[\omega_\Lambda] = [\omega_0]$. In view of Proposition 18 and Proposition 19, $[\omega_{\psi(s_1)}] \neq [\omega_0]$, $\forall s_1 \in]1, \frac{3}{2}]$, and $[\omega_{\psi(s_1)}] \neq [\omega_{\psi(s_2)}]$, $\forall s_1, s_2 \in]1, \frac{5}{4}[\cup]\frac{5}{4}, \frac{3}{2}]$, $s_1 \neq s_2$.
- Therefore, uncountable infinite family of equivalence classes.
- Existence of several distinct equivalent classes implies that there is no state that is at finite distance to all other states.

Proposition 28

For any state, there is at least another state which is at infinite distance.

- This latter property applies to pure and non pure states.

Compact Quantum Metric Space

Rieffel observation: Let commutative compact metric space (X, ρ). Lipschitz semi norm $l(f)$ on $\mathbb{A} := C(X)$. Then, one can define on $\mathcal{S}(\mathbb{A})$ a distance $\rho_l(\omega_1, \omega_2) = \sup(|(\omega_1 - \omega_2)(f)|,\ l(f) \leq 1)$ such that $\lim \rho_l(\omega_n, \omega) = 0 \iff \lim(\omega_n(f) - \omega(f)) = 0,\ \forall f \in \mathbb{A}$.

Definition 29 (Rieffel, Contemp. Math. 2004)

A Compact Quantum Metric Space (CQMS) is a order unit space $\mathbb{A}$ equiped with a seminorm l such that $l(1) = 0$ and the distance defined by

$$d(\omega_1, \omega_2) = \sup\left(|\omega_1 - \omega_2(a)|,\ /\ l(a) \leq 1\right) \quad (14)$$

induced the weak* topology on the state space of $\mathbb{A}$.

- Order unit space: linear sp. of self-adjoint operators on some $\mathcal{H}$ with unit. State notion extend to this space.
- Spectral distance: sup is reached on self-adjoint elements.
- Therefore: unital spectral triple whose spectral distance induces weak* topology on $\mathcal{S}(\mathbb{A})$: CQMS.

The space of pure states - Proof

Proof.

We work with $\bar{\mathcal{A}}$.
To show ω_ψ's are pure and that all pure states are of this kind: show that $\bar{\mathcal{A}}$ is (isometrically) isomorphic to algebra of compact operators $\mathcal{K}(\mathcal{G}_0)$, since by [Kadison, 10.4.4, II, p.750] $\mathcal{K}(\mathcal{G}_0)$ is set of vector states of $\mathcal{G}_0$, actually defined in the proposition.
i) Use GNS representation $\{\pi_m, \mathcal{H}_m\}$ induced by ω_{mn}.
Since $(a^*a)_{pq} = \sum_l \bar{a}_{lp}a_{lq}$, the left kernel N_m of ω_{mn} is the ideal generated by $\{f_{pq}\}_{p\in\mathbb{N},q\in\mathbb{N}/\{m\}}$ so that $\mathcal{H}_m := \bar{\mathcal{A}}/N_m = \mathcal{G}_m$.. As GNS repres. faithful, $\bar{\mathcal{A}}$ is *-isomorphic and so isometrically isomorphic (any injective C* morphism is isometric) to the C^*-algebra $\pi_m(\bar{\mathcal{A}}) \subset \mathcal{B}(\mathcal{H}_m)$.
ii) Let $\mathcal{I}$, the set of finite rank operators on $\mathcal{K}(\mathcal{G}_0)$. For any f_{pq}, $\pi_m(f_{pq}) \in \mathcal{I}$. So any finite rank operator can be written as a finite sum of f_{pq}. Therefore, $\mathcal{I} \subset \pi_m(\bar{\mathcal{A}})$, hence $\bar{\mathcal{I}} := \mathcal{K}(\mathcal{G}_0) \subset \overline{\pi_m(\mathcal{A})} = \pi_m(\bar{\mathcal{A}})$.
iii) Conversely, $\pi_m(\bar{\mathcal{A}}) \subset \mathcal{K}(\mathcal{G}_0)$ (use matrix base to show L_a is Hilbert-Schmidt on $\mathcal{G}_0$, therefore compact. Therefore, one has $\bar{\mathcal{A}} = \mathcal{K}(\mathcal{G}_0)$. The result follows. □

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Technical Lemma - Proof

Proof.

If $||[D,\pi(a)]||_{op}\le 1$, then $||\partial a||_{op}\le\frac{1}{\sqrt{2}}$ and $||\bar\partial a||_{op}\le\frac{1}{\sqrt{2}}$. Use matrix base: for any $\varphi\in\mathcal{H}_0$, $||\partial a\star\varphi||_2^2=2\pi\theta\sum_{m,n}|\sum_p\alpha_{mp}\varphi_{pn}|^2$. From def. of $||\partial a||_{op}$, one get $\sum_{m,n}|\sum_p\alpha_{mp}\varphi_{pn}|^2\le\frac{1}{4\pi\theta}$ for any $\varphi\in\mathcal{H}_0$ with $\sum_{m,n}|\varphi_{mn}|^2=\frac{1}{2\pi\theta}$. Then

$$|\sum_p\alpha_{mp}\varphi_{pn}|\le\frac{1}{2\sqrt{\pi\theta}},\ \forall\varphi\in\mathcal{H}_0,\ ||\varphi||_2=1,\ \forall m,n\in\mathbb{N}\qquad(15)$$

(same for β_{mn}). Now, $|\sum_p\alpha_{mp}\varphi_{pn}|\le\frac{1}{2\sqrt{\pi\theta}}$ true for any $\varphi\in\mathcal{H}_0$ with $||\varphi||_2=1$¿ One can construct $\tilde\varphi$ with $||\tilde\varphi||_2=||\varphi||_2$ via $\alpha_{mp}\tilde\varphi_{pn}=|\alpha_{mp}||\varphi_{pn}|$. Then

$$\sum_p|\alpha_{mp}||\varphi_{pn}|\le\frac{1}{2\sqrt{\pi\theta}},\ \forall\varphi\in\mathcal{H}_0,\ ||\varphi||_2=1,\ \forall m,n\in\mathbb{N}\qquad(16)$$

Notice that (16) implies (15). Similar considerations apply for the β_{mn}'s. The property i) follows.

□

Technical Lemma - Proof II

Proof.

To prove ii): observe that if a is radial, one has $\alpha_{mn}=0$ if $m\neq n+1$ (from matrix base). Then, for any unit vector $\psi\in\mathcal{H}_0$

$$||\partial a\star\psi||_2^2=2\pi\theta\sum_{p,q}|\sum_r\alpha_{pr}\psi_{rq}|^2=2\pi\theta\sum_{p,q}|\alpha_{p,p-1}\psi_{p-1,q}|^2\le\pi\theta\sum_{p,q\in\mathbb{N}}|\psi_{pq}|^2 \quad (17)$$

so that $||\partial a||_{op}^2\le\frac{1}{2}$ showing that a is in the unit ball.

To prove that $\hat{a}(m_0)\in\mathcal{A}$ defines a positive operator of $\mathcal{B}(\mathcal{H})$ for any fixed $m_0\in\mathbb{N}$, show: $<\psi,\pi(\hat{a}(m_0))\psi>\ge 0$, $\forall\psi\in\mathcal{H}$, for any fixed $m_0\in\mathbb{N}$. Set $\psi=(\varphi_1,\varphi_2)$, $\varphi_i\in L^2(\mathbb{R}^2)$, $i=1,2$ and $\varphi_i=\sum_{m,n\in\mathbb{N}}\varphi^i_{mn}f_{mn}$. A matter of standard calculation.

Finally, notice that any positive element $a\in\mathcal{A}_+$ verifies $a^\dagger=a$ so that $(\partial a)^\dagger=\bar{\partial}a$. Therefore $||[D,\pi(a)]||_{op}=\sqrt{2}||\partial a||_{op}$. Now, standard calculation shows that the only non-vanishing coefficients $\hat{\alpha}_{pq}$ in the expansion of $\partial\hat{a}(m_0)$ satisfy $\hat{\alpha}_{p+1,p}=-\frac{1}{\sqrt{2}}$, $0\le p\le m_0$, for any fixed $m_0\in\mathbb{N}$. From the very definition of $||.||_{op}$, one infers that $||\partial\hat{a}(m_0)||_{op}=\frac{1}{\sqrt{2}}$(use for instance (17)). Therefore, one obtains $||[D,\pi(\hat{a}(m_0))]||_{op}=1$ for any $m_0\in\mathbb{N}$. □

Properties of $(\mathcal{S}, \star))$

Theorem 30 (see Gracia-Bondia, Varilly, 1988)

$(\mathcal{S}, \star)$ *is a non unital associative involutive Fréchet algebra with faithful trace and jointly continuous product.*

Proof.

- Associativity and faithfull trace standard
- Continuity of $\star$ in the product topology in $\mathcal{S}$: use estimate $||a \star b||_\infty \leq ||a||_1||b||_1$.
- Then: prove estimates for $x^\alpha \partial^\beta (a \star b)$, $\forall \alpha, \beta \in \mathbb{N}^2$. One get: $\star$ is continuous in $\mathcal{S}$ separately so it is jointly because $\mathcal{S}$ is Fréchet.

□

Properties of $(\mathcal{S},\star))$ - II

- $\star$-product can be extended to other subspaces of $\mathcal{S}'$ (use duality and continuity of $\star$ on $\mathcal{S}$).

- Then, for any $a\in\mathcal{G}_{s,t}$ and $b\in\mathcal{G}_{q,r}$, $b=\sum_{m,n}b_{mn}f_{mn}$, $t+q\ge0$, the sequences $c_{mn}=\sum_p a_{mp}b_{pn}$, $\forall m,n\in\mathbb{N}$ define the functions $c=\sum_{m,n}c_{mn}f_{mn}$, $c\in\mathcal{G}_{s,r}$ [See e.g Gracia-Bondia, Varilly, JMP 1988].

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Properties of $(\mathcal{S},\star))$ - II

- $\star$-product can be extended to other subspaces of $\mathcal{S}'$ (use duality and continuity of $\star$ on $\mathcal{S}$).
- Convenient: Hilbert spaces $\mathcal{S}\subset\mathcal{G}_{s,t}\subset\mathcal{S}'$, $s,t\in\mathbb{R}$,
$\mathcal{G}_{s,t}=\{a=\sum a_{mn}f_{mn}\in\mathcal{S}'\ /\ ||a||^2_{s,t}=\sum_{m,n}\theta^{s+t}(m+\frac{1}{2})^s(n+\frac{1}{2})^t|a_{mn}|^2<\infty\}$
- Then, for any $a\in\mathcal{G}_{s,t}$ and $b\in\mathcal{G}_{q,r}$, $b=\sum_{m,n}b_{mn}f_{mn}$, $t+q\geq0$, the sequences $c_{mn}=\sum_p a_{mp}b_{pn}$, $\forall m,n\in\mathbb{N}$ define the functions $c=\sum_{m,n}c_{mn}f_{mn}$, $c\in\mathcal{G}_{s,r}$ [See e.g Gracia-Bondia, Varilly, JMP 1988].

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Properties of $(\mathcal{S},\star)$) - II

- $\star$-product can be extended to other subspaces of $\mathcal{S}'$ (use duality and continuity of $\star$ on $\mathcal{S}$).
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- Uses: $||a\star b||_{s,r}\le||a||_{s,t}||b||_{q,r},\ t+q\ge0$ and $||a||_{u,v}\le||a||_{s,t}$ if $u\le s$, $v\le t$.
- Then, for any $a\in\mathcal{G}_{s,t}$ and $b\in\mathcal{G}_{q,r}$, $b=\sum_{m,n}b_{mn}f_{mn}$, $t+q\ge0$, the sequences $c_{mn}=\sum_p a_{mp}b_{pn}$, $\forall m,n\in\mathbb{N}$ define the functions $c=\sum_{m,n}c_{mn}f_{mn}$, $c\in\mathcal{G}_{s,r}$ [See e.g Gracia-Bondia, Varilly, JMP 1988].

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Noncommutative Torus - preliminaries

1. THE SPECTRAL DISTANCE
2. MOYAL - BASICS
3. DISTANCE ON MOYAL PLANE
4. DISCUSSION
5. GAUGE THEORIES ON MOYAL SPACES: RESULTS
6. **Noncommutative Torus - preliminaries**
   - basic properties
   - Pure states on noncommutative torus
   - Preliminary results - Spectral distance on NC Torus

The noncommutative torus

Definition 31 (For reviews see, e.g Landi, Gracia-Bondia, Varilly)

$\mathfrak{A}_\theta^2$ universal C*-algebra generated by u_1, u_2 with $u_1 u_2 = e^{i2\pi\theta} u_2 u_1$. Algebra of the noncommutative torus $\mathbb{T}_\theta^2$ is the dense (unital) pre-C* subalgebra of $\mathfrak{A}_\theta^2$ defined by $\mathbb{T}_\theta^2 = \{a = \sum_{i,j\in\mathbb{Z}} a_{ij} u_1^i u_2^j \ / \ \sup_{i,j\in\mathbb{Z}} (1+i^2+j^2)^k |a_{ij}|^2 < \infty\}$.

- Weyl generators defined by $U^M \equiv e^{-i\pi m_1\theta m_2} u_1^{m_1} u_2^{m_2}$, $\forall M = (m_1, m_2) \in \mathbb{Z}^2$. For any $a \in \mathbb{T}_\theta^2$, $a = \sum_{m\in\mathbb{Z}^2} a_M U^M$. Let δ_1 and δ_2: canonical derivations $\delta_a(u_b) = i2\pi u_a \delta_{ab}$, $\forall a, b \in \{1, 2\}$. One has $\delta_b(a^*) = (\delta_b(a))^*$, $\forall b = 1, 2$.

Proposition 32

One has for any $M, N \in \mathbb{Z}^2$, $(U^M)^* = U^{-M}$, $U^M U^N = \sigma(M,N) U^{M+N}$ *where the commutation factor* $\sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ *satisfies*

$$\sigma(M+N,P) = \sigma(M,P)\sigma(N,P),\ \sigma(M,N+P) = \sigma(M,N)\sigma(M,P),\ \forall M,N,P \in \mathbb{Z}^2$$

$$\sigma(M, \pm M) = 1,\ \forall M \in \mathbb{Z}^2$$

$$\delta_a(U^M) = i2\pi m_a U^M,\ \forall a = 1,2,\ \forall M \in \mathbb{Z}^2$$

The noncommutative torus

- Let τ be tracial state:
  For any $a=\sum_{M\in\mathbb{Z}^2} a_M U^M \in \mathbb{T}^2_\theta$, $\tau:\mathbb{T}^2_\theta \to \mathbb{C}$, $\tau(a)=a_{0,0}$.

The noncommutative torus

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- $\mathcal{H}_\tau$: GNS Hilbert space (completion of $\mathbb{T}^2_\theta$ in the Hilbert norm induced by $< a,b > \equiv \tau(a^*b)$). One has $\tau(\delta_b(a))=0$, $\forall b=1,2$.

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- The even real spectral triple:

  $$(\mathbb{T}^2_\theta, \mathcal{H}, D; J, \Gamma)$$

  $\mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^2$. One has $\delta_b^\dagger = -\delta_b$, $\forall b = 1, 2$, in view of
  $< \delta_b(a), c >= \tau((\delta_b(a)^* c) = \tau(\delta_b(a^*)c) = -\tau(a^*\delta_b(c)) = - < a, \delta_b(c) >$ for any $b = 1, 2$ and $\delta_b(a^*) = (\delta_b(a))^*$.

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- Define $\delta = \delta_1 + i\delta_2$ and $\bar{\delta} = \delta_1 - i\delta_2$. D: unbounded self-adjoint Dirac operator $D = -i \sum_{b=1}^2 \delta_b \otimes \sigma^b$, densely defined on $\mathrm{Dom}(D) = (\mathbb{T}^2_\theta \otimes \mathbb{C}^2) \subset \mathcal{H}$.
  $$D = -i \begin{pmatrix} 0 & \delta \\ \bar{\delta} & 0 \end{pmatrix}$$

The noncommutative torus

- Let τ be tracial state:
  For any $a=\sum_{M\in\mathbb{Z}^2}a_MU^M\in\mathbb{T}^2_\theta$, $\tau:\mathbb{T}^2_\theta\to\mathbb{C}$, $\tau(a)=a_{0,0}$.
- $\mathcal{H}_\tau$: GNS Hilbert space (completion of $\mathbb{T}^2_\theta$ in the Hilbert norm induced by $<a,b>\equiv\tau(a^*b)$). One has $\tau(\delta_b(a))=0$, $\forall b=1,2$.
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- Define $\delta=\delta_1+i\delta_2$ and $\bar\delta=\delta_1-i\delta_2$. D: unbounded self-adjoint Dirac operator $D=-i\sum_{b=1}^2\delta_b\otimes\sigma^b$, densely defined on $\text{Dom}(D)=(\mathbb{T}^2_\theta\otimes\mathbb{C}^2)\subset\mathcal{H}$.
$$D=-i\begin{pmatrix}0&\delta\\ \bar\delta&0\end{pmatrix}$$
- Faithfull representation $\pi:\mathbb{T}^2_\theta\to\mathcal{B}(\mathcal{H}):\pi(a)=L(a)\otimes\mathbb{I}_2$, $\pi(a)\psi=(a\psi_1,a\psi_2)$, $\psi=(\psi_1,\psi_2)\in\mathcal{H}$, $\forall a\in\mathbb{T}^2_\theta$. $L(a)$: left multiplication operator by any $a\in\mathbb{T}^2_\theta$. $\pi(a)$ and $[D,\pi(a)]$ bounded on $\mathcal{H}$ for any $\mathbb{T}^2_\theta$.
$$[D,\pi(a)]\psi=-i(L(\delta_b(a))\otimes\sigma^b)\psi=-i\begin{pmatrix}L(\delta(a))&0\\0&L(\bar\delta(a))\end{pmatrix}\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix}\qquad(18)$$

Pure states on noncommutative torus

- Classification of the pure states in the irrational case is lacking.

Pure states on noncommutative torus

- Classification of the pure states in the irrational case is lacking.
- Consider rational case: $\theta = \frac{p}{q}$, $p < q$, p and q relatively prime, $q \neq 1$. Set $\mathbb{T}^2_{\frac{p}{q}} \equiv T_{p/q}$ [see e.g Connes, Landi, Rieffel]. Unitary equivalence classes of irreps. $T_{p/q}$ classified by a torus parametrized by (α, β). Irreps. given by $\pi_{\alpha,\beta} : T_{p/q} \to \mathbb{C}^q$, $\alpha, \beta \in \mathbb{C}$ unitaries and $\pi_{\alpha,\beta}(u_1), \pi_{\alpha,\beta}(u_2) \in \mathbb{M}_q(\mathbb{C})$ are the usual clock and shift matrices in the basis defined by $\{e_k = \beta^{-k/q} u_2^k e_0\}$, $\forall k \in \{0, 1, ..., q-1\}$ and $u_1 e_0 = \alpha^{1/q} e_0$.

Proposition 33

The set of pure states of the rational noncommutative torus is exactly the set of vector states $\omega^{\psi}_{\alpha,\beta} : T_{p/q} \to \mathbb{C}$

$$\omega^{\psi}_{\alpha,\beta}(a) = (\psi, \pi_{\alpha,\beta}(a)\psi),\ \forall \psi \in \mathbb{C}^q,\ ||\psi|| = 1 \qquad (19)$$

where ψ is given up to an overall phase. The pure states are then classified by a bundle over a commutative torus parametrized by (α, β) with fiber $P(\mathbb{C}^q)$.

Proof.

By standard results on C*-algebras, any irrep. $(\pi_{\alpha,\beta}, \mathbb{C}^q)$ is unitarily equivalent to the GNS representation $(\omega_\psi, \pi_{\alpha,\beta})$ for any $\psi \in \mathbb{C}^q$. Then, the ω_ψ are pure states. Write now $\omega^{\psi}_{\alpha,\beta}(a) = (\psi, \pi_{\alpha,\beta}(a)\psi)$ for any $a \in T_{p/q}$. □

Preliminary results - Spectral distance on NC Torus

- One has

Lemma 34

Set $\delta(a)=\sum_{N\in\mathbb{Z}^2}\alpha_N U^N$. *One has* $\alpha_N=i2\pi(n_1+in_2)a_N,\ \forall N=(n_1,n_2)\in\mathbb{Z}^2$.
i) For any a in the unit ball, $||[D,\pi(a)||_{op}\le 1$ *implies* $|\alpha_N|\le 1,\ \forall N\in\mathbb{Z}^2$. *Similar results hold for* $\bar{\delta}(a)$.
ii) The elements $\hat{a}^M\equiv\frac{U^M}{2\pi(m_1+im_2)}$ *verify* $||[D,\pi(\hat{a}^M)||_{op}=1,\ \forall M=(m_1,m_2)\in\mathbb{Z}^2$, $M\neq(0,0)$

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i) For any a in the unit ball, $||[D,\pi(a)||_{op}\le 1$ implies $|\alpha_N|\le 1$, $\forall N\in\mathbb{Z}^2$. Similar results hold for $\bar{\delta}(a)$.
ii) The elements $\hat{a}^M\equiv\frac{U^M}{2\pi(m_1+im_2)}$ verify $||[D,\pi(\hat{a}^M)||_{op}=1$, $\forall M=(m_1,m_2)\in\mathbb{Z}^2$, $M\neq(0,0)$

- Indeed

Proof.

The relation involving α_N obvious. Then, $||[D,\pi(a)||_{op}\le 1$ is equivalent to $||\delta(a)||_{op}\le 1$ and $||\bar{\delta}(a)||_{op}\le 1$ in view of (18). For any $a\in\mathfrak{A}^2_\theta$ and any unit $\psi=\sum_{N\in\mathbb{Z}^2}\psi_N U^N\in\mathcal{H}_\tau$, one has $||\delta(a)\psi||^2=\sum_{N\in\mathbb{Z}^2}|\sum_{P\in\mathbb{Z}^2}\alpha_P\psi_{N-P}\sigma(P,N)|^2$. Then $||\delta(a)||_{op}\le 1$ implies $|\sum_{P\in\mathbb{Z}^2}\alpha_P\psi_{N-P}\sigma(P,N)|\le 1$, for any $N\in\mathbb{Z}^2$ and any unit $\psi\in\mathcal{H}_\tau$. By a straightforward adaptation of the proof carried out for ii) of Lemma **??**, this implies $|\alpha_M|\le 1$, $\forall M\in\mathbb{Z}^2$. This proves ii). Finally, iii) stems simply from an elementary calculation. □

Preliminary results - Spectral distance on NC Torus

- The following proerty holds

Proposition 35

Let the familly of unit vectors $\Phi_M=(\frac{1+U^M}{\sqrt{2}},0)\in\mathcal{H}$, $\forall M\in\mathbb{Z}^2$, $M\ne(0,0)$ *generating the family of vector states of* $\mathbb{T}^2_\theta$

$$\omega_{\Phi_M}:\mathbb{T}^2_\theta\to\mathbb{C},\ \omega_{\Phi_M}(a)\equiv(\Phi_M,\pi(a)\Phi_M)_{\mathcal{H}}=\frac{1}{2}<(1+U^M),(a+aU^M)> \quad (20)$$

The spectral distance between any state ω_{Φ_M} *and the tracial state is*

$$d(\omega_{\Phi_M},\tau)=\frac{1}{2\pi|m_1+im_2|},\ \forall M=(m_1,m_2)\in\mathbb{Z}^2,\ M\ne(0,0) \quad (21)$$

Preliminary results - Spectral distance on NC Torus

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$$\omega_{\Phi_M} : \mathbb{T}^2_\theta \to \mathbb{C},\ \omega_{\Phi_M}(a) \equiv (\Phi_M, \pi(a)\Phi_M)_{\mathcal{H}} = \frac{1}{2} < (1+U^M), (a+aU^M) > \quad (20)$$

The spectral distance between any state ω_{Φ_M} *and the tracial state is*

$$d(\omega_{\Phi_M}, \tau) = \frac{1}{2\pi|m_1 + im_2|},\ \forall M = (m_1, m_2) \in \mathbb{Z}^2,\ M \neq (0,0) \quad (21)$$

- Sketch

Proof.

Set $a = \sum_{N \in \mathbb{Z}^2} a_N U^N$. Using Proposition 32 yields $\omega_{\Phi_M}(a) = \tau(a) + \frac{1}{2}(a_M + a_{-M})$. This, combined with Lemma 34 yields $d(\omega_{\Phi_M}, \tau) \leq \frac{1}{2\pi|m_1+im_2|}$. Upper bound obviously saturated by the element $\hat{a}^M$ of iii) of Lemma 34 which belongs to the unit ball. □