Distribution of the time at which $N$ vicious walkers reach their maximal height

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We study the extreme statistics of $N$ nonintersecting Brownian motions (vicious walkers) over a unit time interval in one dimension. Using path-integral techniques we compute exactly the joint distribution of the maximum $M$ and of the time $\tau_M$ at which this maximum is reached. We focus in particular on nonintersecting Brownian bridges (“watermelons without wall”) and nonintersecting Brownian excursions (“watermelons with a wall”). We discuss in detail the relationships between such vicious walkers models in watermelon configurations and stochastic growth models in curved geometry on the one hand and the directed polymer in a disordered medium (DPRM) with one free end point on the other hand. We also check our results using numerical simulations of Dyson’s Brownian motion and confront them with numerical simulations of the polynuclear growth model (PNG) and of a model of DPRM on a discrete lattice. Some of the results presented here were announced in a recent letter [J. Rambeau and G. Schehr, Europhys. Lett. 91, 60006 (2010)].

I. INTRODUCTION AND MOTIVATIONS

Extreme value statistics (EVS) is at the heart of optimization problems and, as such, it plays a crucial role in the theory of complex and disordered systems [1,2]. For instance, to characterize the thermodynamical properties of a disordered or glassy system at low temperature, one is often interested in computing its ground state, that is, the configuration with lowest energy. Similarly, the low temperature dynamics of such systems is determined, at large times, by the largest energy barriers of the underlying free energy landscape. Thus the distribution of the time at which this maximum is reached (see Fig. 4): 

\[ x_1(\tau) < \cdots < x_N(\tau), \quad \forall \tau \in [0,1], \]

on the unit time interval, $\tau \in [0,1]$. We consider different types of configurations of such vicious walkers, which are conveniently represented in the $(x, t)$ plane. In the first case, called “watermelons,” all the walkers start, at time $\tau = 0$, and end, at time $\tau = 1$, at the origin (see Sec. II E). They thus correspond to nonintersecting Brownian bridges. In the second case, we consider such watermelons with the additional constraint that the positions of the walkers have to stay positive (see Sec. II G): We call these configurations “watermelons with a wall” and they thus correspond to nonintersecting Brownian excursions. Finally, we also consider “stars” configuration, where the end points of the BMs in $\tau = 1$ are free. They correspond to nonintersecting free BMs (see Sec. II F).

Motivated by extreme value questions, we compute here the joint distribution $P_N(M, \tau_M)$ of the maximal height and the time at which this maximum is reached (see Fig. 4):

\[ M = \max_{0 \leq \tau \leq 1} x_N(\tau), \quad x_N(\tau_M) = M. \]

The (marginal) distribution of $M$ has recently been studied by several authors [28–34], while we have announced exact results for this joint distribution in the two first cases (for bridges and excursions) in a recent letter [35]. The goal of this paper is to give a detailed account of the method and the

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computations leading to these results. On this route we also provide several new results, including, for instance, results for the star configuration.

In the physics literature, such nonintersecting BMs were first introduced by de Gennes in the context of fibrous polymers [36]. They were then widely studied after the seminal work of Fisher [37], who named them “vicious walkers,” as the process is killed if two paths cross each other. These models have indeed found many applications in statistical physics, ranging from wetting and melting transitions [37], commensurate-incommensurate transitions [38], and networks of polymers [39] to persistence properties in nonequilibrium systems [40].

Vicious walkers have also very interesting connections with random matrix theory (RMT), in particular through Dyson’s BM [41]. For example, for wettermanol configurations, it can be shown that the positions of the \( N \) random walkers at a fixed time \( \tau \), correctly scaled by a \( \tau \)-dependent factor, are distributed like the \( N \times N \) Hermitian matrices of the Gaussian unitary ensemble of RMT corresponding to \( \beta = 2 \) [42]. If one denotes by \( P_{\text{join}}(x,\tau) \equiv P_{\text{point}}(x_1, \ldots, x_N, \tau) \) the joint distribution of the positions of the walkers at a given time \( \tau \), in the wettermanol configuration, one has indeed (see, for instance, Ref. [30] for a rather straightforward derivation of this result)

\[
P_{\text{join}}(x,\tau) = Z_N^{-1} \sigma(\tau)^{-N^2} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 e^{-\frac{x_i^2 + x_j^2}{2\sigma^2(\tau)}},
\]

where \( Z_N \) is a normalization constant and \( \sigma(\tau) = \sqrt{\tau(1 - \tau)} \) and where we use the notation \( x^2 = \sum_{i=1}^{N} x_{i}^2 \). This means in particular for the top path that \( x_N(\tau)/\sqrt{\tau(1 - \tau)} \) is, at fixed \( \tau \), distributed like the largest eigenvalue of Gaussian unitary ensemble (GUE) random matrices, which means that in the large \( N \) limit one has [43]

\[
\frac{x_N(\tau)}{\sqrt{2\sigma(\tau)}} = \sqrt{2N} + \frac{1}{\sqrt{2}} N^{-1/6} \chi_2, \tag{4}
\]

where \( \chi_2 \) is distributed according to the Tracy-Widom (TW) distribution for \( \beta = 2 \), \( \mathcal{F}_2 \), namely, \( \Pr[\chi_2 \leq x] = \mathcal{F}_2(x) \). In the case of wettermanol with a wall, the positions of the random walkers are instead related to the eigenvalues of Wishart matrices [30,44] and the fluctuations of the top path \( x_N(\tau) \) are again described by \( \mathcal{F}_2(x) \).

Yet another reason why there is currently a rekindled interest in vicious walkers problems is because of their connection with stochastic growth processes in the Kardar-Parisi-Zhang (KPZ) universality class [45,46] in 1 + 1 dimensions. This connection is believed to hold for any systems belonging to the KPZ universality class [47], and it can be rigorously shown on one particular model of stochastic growth, the so-called polynuclear growth model (PNG) [48,49]. It is defined as follows (see Fig. 1). At time \( t = 0 \) a single island starts spreading on a flat substrate at the origin \( x = 0 \) with unit velocity. Seeds of negligible size then nucleate randomly at a constant rate \( \rho = 2 \) per unit length and unit time and then grow laterally also at unit velocity. When two islands on the same layer meet they coalesce. Meanwhile, nucleations continuously generate additional layers. In the flat geometry nucleations can occur at any point \( x \), while in the droplet geometry, which we focus on, nucleations only occur above previously formed layers. Therefore, for the droplet geometry, denoting by \( h_{\text{drop}}(x,t) \) the height of the interface at point \( x \) and time \( t \), one has \( h_{\text{drop}}(x,t) = 0 \) for \( |x| > t \). On the other hand, in the long time limit, the profile for \( |x| \leq t \) becomes droplet like \( h_{\text{drop}}(x,t) \sim 2t \left( \sqrt{1 - (x/t)^2} \right) \), but there remain height fluctuations around this mean value [see Fig. 2(a)]. The standard way to characterize these fluctuations is to look at the width of the interface \( W(t) = \left( \langle h_{\text{drop}}(x,t) - \langle h_{\text{drop}}(x,t) \rangle \rangle^2 \right)^{1/2} \sim t^\gamma \) with universal exponents \( \gamma = \frac{1}{3} \) and \( \zeta = \frac{3}{4} \) [51,52], in agreement with the fact that this model belongs to the KPZ universality class. More recently, it was shown for several models belonging to the KPZ universality class [53–58] that, in the growth regime \( t_0 \ll t \ll L \) (where \( t_0 \) is a microscopic time scale) universality extends far beyond the exponents \( \gamma \) and \( \zeta \) but also applies to full distribution functions of physical observables. In particular, the scaled cumulative distribution of the height field at a given point coincides with the TW distribution \( \mathcal{F}_\beta \) with \( \beta = 2 \) (\( \beta = 1 \)) for the curved geometry (the flat geometry), which describes the edge of the spectrum of random matrices in the Gaussian unitary ensemble (the Gaussian orthogonal ensemble) [43]. Height fluctuations were measured in experiments, both in planar [59] and more recently in curved geometry in the electroconvection of nematic liquid crystals [60] and a good quantitative agreement with TW distributions was found.

Although the relation with random matrices was initially achieved [53] through the longest increasing subsequence of random permutations [58,61], using in particular the results of the seminal paper by Baik, Deift, and Johansson [62], it was then realized that the PNG model in the droplet geometry is actually directly related to the vicious walkers problem in the watermelon configuration [54]. This was shown through an extension of the PNG model to the so-called multilayer PNG model, where nonintersecting paths naturally appear. Indeed, one can show that the fluctuations of the height field of the PNG model in the droplet geometry \( h_{\text{drop}}(x,t) \) are related, in the large time limit, to the fluctuations of the top path \( x_N(\tau) \) for the vicious walkers problem in the watermelon geometry in the large \( N \) limit. This mapping, for \( N, t \gg 1 \), reads [54,63]

\[
\frac{h_{\text{drop}}(u^2, t)}{\sqrt{t}} - 2t = 2(x_N(u^2) + \frac{u^2}{2N^2}) - N^{\frac{1}{2}} = A_2(u) - u^2, \tag{5}
\]

where \( A_2(u) \) is the Airy2 process [54] which is a stationary and non-Markovian process.\footnote{Note that in Eq. (5) we have corrected a missing factor of 2 in formula (1) of Ref. [35]} In particular, \( \Pr[A_2(0) \leq x] \)
$\mathcal{F}_2(x)$, which is consistent with Eq. (4). Hence, from Eq. (5) $x_N$ and $\tau$ map onto $h$ and $x$ in the growth model while $N$ plays essentially the role of $t$ (to make the correspondence between $N$ and $t$ exact one has to consider a watermelon configuration in the interval $\tau \in [0,N]$). To characterize the fluctuations of the height profile in the droplet geometry beyond the standard roughness $W_1(t)$ it is natural to consider the maximal height $M$ and its position $X_M$ [35] [see Fig. 2(a)]. According to KPZ scaling, one expects $M - 2t \sim \mathcal{O}(t^{1/3})$ while $X_M \sim t^{2/3}$. On the other hand, from Eq. (5), the joint distribution $P_t(M,X_M)$ of $M$ and $X_M$ can be written as

$$ P_t(M,X_M) \sim t^{-1} P_{\text{Airy}}((M-2t)t^{-1},X_M t^{-2}), \quad (6) $$

where $P_{\text{Airy}}(y,x)$ is the joint distribution of the maximum $y$ and its position $x$ for the process $\mathcal{A}_2(u) - u^2$. Finally, the relation (5) also gives us some interesting information for the vicious walkers problem. In the large $N$ limit, one has indeed from Eq. (5) that the joint distribution $P_N(M,\tau_M)$ for the watermelon configuration is also given by $P_{\text{Airy}}(y,x)$. One has indeed, for $N \gg 1$,

$$ P_N(M,\tau_M) \sim 4N^{\frac{3}{2}} P_{\text{Airy}} \left[ 2(M - \sqrt{N}) N^{\frac{1}{2}}, 2 \left( \tau_M - \frac{1}{2} \right) N^{\frac{3}{2}} \right], \quad (7) $$

and therefore Eqs. (6) and (7) show that one can obtain the distribution of $P_t(M,X_M)$ for the growth model from the large $N$ limit of the joint distribution $P_N(M,\tau_M)$, which we compute here for any finite $N$.

It is also well known that stochastic growth models in the KPZ universality class can be mapped onto the model of the directed polymer in a disordered medium (DPRM) [46,64]. This is also the case of the PNG model and to make this connection as clear as possible, we consider a discrete version of the PNG model introduced by Johansson [65]. It is a growth model with discrete space and discrete time. The height function $h(x,t)$ is now an integer value $h(x,t) \in \mathbb{N}$, while $x \in \mathbb{Z}$ and $t \in \mathbb{N}$. The dynamics is defined as follows:

$$ h(x,t+1) = \max[h(x-1,t), h(x,t), h(x+1,t)] + \omega(x,t+1), \quad (8) $$

where the first term reproduces the lateral expansion of the islands (and also their coalescence when two of them meet) in the continuous PNG model, while $\omega(x,t) \in \mathbb{N}$ is a random variable, distributed independently from site to site, which corresponds to the random nucleations (see Fig. 1).

In the droplet geometry, one has $\omega(x,t) = 0$ if $|x| > t$, while this constraint is removed in the flat geometry. To understand better the connection between this model (8) and a model of directed polymer, we perform a rotation of the axis $(x,t)$ (Fig. 3) and define

$$ \omega(i-j, i+j-1) \equiv w(i,j). \quad (9) $$

One can then simply check (for instance, by induction on $t$) that the height field $h_{\text{drop}}(x,t)$ evolving with Eq. (8) in the droplet geometry is given by

$$ h_{\text{drop}}(X,t) = \max \sum_{\mathcal{C} \subseteq \mathcal{P}_t(0,X)} w(i,j), \quad (10) $$

where $\mathcal{P}_t(0,X)$ is the ensemble of directed paths of length $t$, starting in $x = 0$ and terminating in $X$ (Fig. 3). This formula (10) establishes a direct link between the PNG model and a

![FIG. 2. (Color online) (a) Hight profile $h_{\text{drop}}(x,t)$ at fixed time $t$, as a function of $x$ for the PNG model in the droplet geometry. $X_M$ is the position at which the maximal hight $M$ is reached. (b) Directed polymer with one free end. Here $M$ corresponds to the energy of the optimal polymer, and $M - 2t \sim \mathcal{O}(t^{1/3})$ while $X_M \sim t^{2/3}$ corresponds to the transverse coordinate of the end point of this optimal polymer.](image)
model of directed polymer. Indeed, the height field \( h_{\text{drop}}(X,t) \) corresponds to the energy of the optimal polymer (which is here the polymer with the highest energy) with both fixed ends in 0 and \( X \) and where the random energies, on each site, are given by \( u(i,j) \). The aforementioned result for the distribution of \( h_{\text{drop}}(X,t) \) shows that the fluctuations of the energy of the optimal polymer are described in this case by \( F_2 \), the TW distribution for \( \beta = 2 \). The study of this distribution, for the DPRM in continuum space, has recently been the subject of several works, both in physics [66] and in mathematics [67].

Similarly, one can also write the height field \( h_{\text{flat}}(0,t) \) in the flat geometry (here, of course, the fluctuations of the height field are invariant by a translation along the \( x \) axis). One has indeed

\[
h_{\text{flat}}(0,t) = \max_{-t \leq X \leq t} \left\{ \max_{c \in \mathbb{P}(X,0)} \left[ \sum_{(i,j) \in c} u(i,j) \right] \right\}, \tag{11}
\]

where \( \mathbb{P}(X,0) \) is the ensemble of directed paths of length \( t \), starting in \( X \) and ending in 0 (see Fig. 3). Given that these two ensembles \( \mathbb{P}_d(X,0) \) and \( \mathbb{P}_0(X,0) \) are obviously similar (Fig. 3), these two equations (10) and (11) make it possible to write [68]

\[
h_{\text{flat}}(0,t) = \max_{-t \leq X \leq t} h_{\text{drop}}(X,t). \tag{12}
\]

For the directed polymer, \( h_{\text{flat}}(0,t) \) corresponds to the energy of the optimal polymer, of length \( t \) and with one free end. From that Eq. (12), one sees that the joint distribution of \( M \) and \( \tau_M \) in the vicious walkers problem corresponds to the joint distribution of the energy \( M \) and the transverse coordinate \( X_M \) of the free end of the optimal polymer. From the aforementioned results for the distribution of \( h_{\text{flat}}(0,t) \), we also conclude that the cumulative distribution of \( M \) is given by \( F_1 \), the TW distribution for \( \beta = 1 \). Although this result was obtained rather indirectly using this relation (12), it was recently shown directly by computing the cumulative distribution of the maximal height of \( N \) nonintersecting excursions in the limit \( N \to \infty \), in Ref. [34]. We conclude this paragraph about the DPRM by noticing that this model, with one free end, has been widely studied in the context of disordered elastic systems [69], in particular, using the approximation of the so-called “toy model” where the Airy_{2} process \( A_{2}^{a}(\tau) \) in (5) is replaced with a BM [2,69]. A recent discussion of the toy model and its applications to the DPRM can be found in Ref. [70]. We also mention that physical observables related to \( X_M \) have been studied for the DPRM in different geometry, such as the winding number of the optimal polymer on a cylinder [71].

The paper is organized as follows. In Sec. II we present in detail the method to compute the joint distribution of \( M \) and \( \tau_M \) for \( N = 2 \) nonintersecting BMs. We show results for bridges, excursions, and stars. In Sec. III we extend this approach for any number \( N \) of vicious walkers both for bridges and excursions. We then confront, in Sec. IV, our analytical results to numerical simulations, which were obtained by simulating Dyson’s BM, before we conclude in Sec. V. Some technical details have been left in the appendices.

### II. DISTRIBUTION OF THE POSITION OF THE MAXIMUM FOR TWO VICIOUS BROWNIAN PATHS

#### A. Introduction

Our purpose is to study \( N \) simultaneous BMs \( x_1(\tau), x_2(\tau), \ldots, x_N(\tau) \), subjected to the condition that they do not cross each other. In the literature, such a system has been studied first in the continuous case by de Gennes [36], and in the discrete case by Fisher [37]. He named this kind of system vicious walkers, as the process is killed if two paths cross each others. The name “vicious Brownian paths” refers in the following to Brownian paths under the condition that they do not cross each other.

In this section, we focus on the case \( N = 2 \). The two vicious Brownian paths start at \( x_1(0) = x_2(0) = 0 \) and obey the noncrossing condition \( x_1(\tau) < x_2(\tau) \) for \( 0 < \tau < 1 \). We examine three different geometries:

1. (i) periodic boundary conditions where we consider nonintersecting bridges, also called “watermelon” configurations (see Fig. 4), treated in Sec. II E;
2. (ii) free boundary conditions where we consider nonintersecting free BMs, also called “stars” configurations (see Fig. 6), treated in Sec. II F;
3. (iii) periodic boundary conditions, plus a positivity constraint where we consider nonintersecting excursions, also called “watermelons with a wall” configurations (see Fig. 8), treated in Sec. II G.

We compute, for each case, the joint distribution of the couple \( (M, \tau_M) \) which are respectively the maximum \( M \) of the uppermost vicious Brownian path, and its position \( \tau_M \): \( M = \max_{0 < \tau < 1} x_2(\tau) = x_2(\tau_M) \). Integrating over \( M \), we find the probability distribution function (PDF) of the time to reach the maximum \( \tau_M \).

This computation is based on a path integral approach, using the link between the problem of vicious Brownian paths and the quantum mechanics of fermions. This is described in Sec. II B. Then we present, in Sec. II C, the method to compute the joint probability distribution function of the maximum and the time to reach it for vicious Brownian paths, with general boundary conditions. An emphasis is put on the regularization scheme, essential to circumvent the basic problem of BM, which is continuous both in space and in time: You cannot force it to be in a point without revisiting it infinitely many times immediately after. At this stage, general formulas are obtained, which are used directly in Secs. II E, II F, and II G. The extension to a general \( N \) number of paths will be treated straightforwardly in Sec. III.

#### B. Path integral approach: Treating the vicious Brownian paths as fermions

Let us consider two BMs, obeying the Langevin equations

\[
\frac{dx_j(\tau)}{d\tau} = \xi_j(\tau), \quad (j = 1, 2), \tag{13}
\]

where \( \xi_j(\tau) \)'s are independent and identical Gaussian white noises of zero mean, that is, \( \langle \xi_j(\tau) \rangle = 0 \) and \( \langle \xi_j(\tau) \xi_j(\tau') \rangle = \delta_{\tau,\tau'} \). Taking into account the noncrossing condition, the probability that the two paths end at \( \{x_1(t_0) \)}
where the subscript “2” in the notation \( p_2 \) means that we treat the case of \( N = 2 \) Brownian paths. The exponential terms are the weight of each Brownian path, and the Heaviside \( \theta \) function ensures that the paths do not cross. In this section, bold letters denote vectors with two components, such as, for example, the states at a given time of the two Brownian paths \( x(\tau) = [x_1(\tau), x_2(\tau)] \) or the starting points \( a = (a_1, a_2) \). In the previous formula (14), \( z \) is a normalization constant, such that

\[
\int_{-\infty}^{+\infty} dB_2 \int_{-\infty}^{+\infty} dB_1 \ p_2(b, t_b | a, t_a) = 1. \tag{15}
\]

Notice that the integration is ordered, and we write it shortly as

\[
\int_{\text{ord}} dB = \int_{-\infty}^{+\infty} dB_2 \int_{-\infty}^{+\infty} dB_1.
\]

We recognize in formula (14) the path integral representation of the quantum propagator of two identical free particles, subjected not to cross each other. In one dimension, this can be implemented by requiring that the two particles are fermions [30,36]. Therefore the probability \( p_2(b, t_b | a, t_a) \) reduces to the quantum propagator of two identical fermions in one dimension,

\[
p_2(b, t_b | a, t_a) = \langle b | e^{-(t_b-t_a)H_{\text{free}}} | a \rangle, \tag{16}
\]

with \( H_{\text{free}} \) the total Hamiltonian of the two-particle system

\[
H_{\text{free}} = H_0^{(1)} \otimes I^{(2)} + I^{(1)} \otimes H_0^{(2)}, \quad I^{(j)} \text{ being the identity operator acting in the Hilbert space of the } j\text{th particle.}
\]

The one-particle Hamiltonians are \( H_0^{(j)} = -\frac{1}{2} \partial_x^2 \) (for \( j = 1, 2 \)) in a position representation. The states \( |a_1, a_2\rangle \) (and \( |b_1, b_2\rangle \)) are the tensorial product of the one-particle eigenstates of position \( |a_1, a_2\rangle = |a_1\rangle \otimes |a_2\rangle \), where each state \( |a_j\rangle \) obeys the eigenvalue equation \( X^{(j)} |a_j\rangle = a_j |a_j\rangle \), with \( X^{(j)} \) the position operator acting in the Hilbert space of the \( j \)th particle and \( a_j \) the position of this particle.

In our calculations we deal with more general Hamiltonians, in the same way as the sum of individual one-particle Hamiltonians \( H = H_0^{(1)} \otimes I^{(2)} + I^{(1)} \otimes H_0^{(2)} \) (no interaction between the two particles) and with \( H_0^{(1)}(x, p) = H_0^{(2)}(x, p) \), identical particles. Thus we treat systems of two identical indistinguishable particles, so that one can write a two-particle eigenstate of the total Hamiltonian \( H \) as an antisymmetric product of the one-particle eigenstates of \( H^{(j)} \). Explicitly, if \( |E_1\rangle \) and \( |E_2\rangle \) are two eigenstates of the one-particle Hamiltonian \( H^{(j)} \), with energies \( E_1 \) and \( E_2 \), then the eigenstate of the two-particle system with energy \( E = E_1 + E_2 \)

\[
|E\rangle = \frac{1}{\sqrt{2}}( |E_1\rangle \otimes |E_2\rangle - |E_2\rangle \otimes |E_1\rangle ). \tag{17}
\]

Projecting onto the position basis \( |x\rangle = |x_1\rangle \otimes |x_2\rangle \), and using the general notation \( \Phi_{E_1}(x_{j}) = \langle x_{j} | E_{j} \rangle \) (lower-case letter \( \Phi \) for one-particle wave function), with \( \Phi_{E}(x) = \langle x | E \rangle \) (capital letter \( \Phi \) for the two-particle wave function), the usual Slater determinant is as follows:

\[
\Phi_{E}(x_1, x_2) = \frac{1}{\sqrt{2!}} \left[ \Phi_{E_1}(x_1) \Phi_{E_2}(x_2) - \Phi_{E_2}(x_1) \Phi_{E_1}(x_2) \right] = \frac{1}{\sqrt{2!}} \det \Phi_{E}(x_j). \tag{18}
\]

Hence, inserting the closure relation \( \int e^{-\int_{t_a}^{t_b} [X(t)] dt} \prod_{t_a < \tau < t_b} \theta (x_2(\tau) - x_1(\tau)) \)

\[
= \int \int e^{-\int_{t_a}^{t_b} [X(t)] dt} \prod_{t_a < \tau < t_b} \theta (x_2(\tau) - x_1(\tau))
\]

for one-particle wave function), with \( \Phi_{E}(x) = \langle x | E \rangle \) (capital letter \( \Phi \) for the two-particle wave function), the usual Slater determinant is as follows:

\[
\Phi_{E}(x_1, x_2) = \frac{1}{\sqrt{2!}} \left[ \Phi_{E_1}(x_1) \Phi_{E_2}(x_2) - \Phi_{E_2}(x_1) \Phi_{E_1}(x_2) \right] = \frac{1}{\sqrt{2!}} \det \Phi_{E}(x_j). \tag{18}
\]

This result can also be found via the Karlin-McGregor formula [72]. It states that the probability of propagating without crossings is nothing but the determinant formed from one-particle propagators (named \( p_1 \)):

\[
p_2(b, t_b | a, t_a) = \det_{t_a < j < t_b} p_1(b_j, t_b | a_j, t_a). \tag{20}
\]

The discrete version of this formula (20) is given by the Lindstr¨om-Gessel-Viennot (LGV) theorem [73]. The basic idea behind this formula is nothing but the method of images. To recover our quantum mechanical expression obtained by considering fermions (19), one has to express the one-particle propagator as

\[
p_1(b_j, t_b | a_j, t_a) = \sum_{E} \Phi_{E}(b_j) \Phi_{E}^{*}(a_j) e^{-E(t_b-t_a)}/Z_{E}(a_j, b_j), \tag{21}
\]

and use the Cauchy-Binet identity (B7).

### C. Joint probability distribution of position and time

The method to compute the distribution of the maximum and the time at which it is reached can be described as follows. Let us consider the process \( \{x_1(\tau), x_2(\tau)\} \) of such Brownian paths with the nonintersecting condition with \( t_a = 0 \) and \( t_b = T \). Keeping the same notations, the starting points are \( x_1(0) = a_1 \) and \( x_2(0) = a_2 \), or \( x(0) = a \), and the end points are \( x_1(T) = b_1 \) and \( x_2(T) = b_2 \), or \( x(T) = b \). First we consider \( a \) and \( b \) as fixed. Given a set of points \( y = (y_1, y_2) \), and an intermediate time \( \tau_{\text{cut}} \), such as \( 0 \leq \tau_{\text{cut}} \leq T \), we ask what is the probability that \( x_1(\tau_{\text{cut}}) \in [y_1, y_1 + dy_1] \) and \( x_2(\tau_{\text{cut}}) \in [y_2, y_2 + dy_2] \)? In other terms, we seek \( Q_2(y, \tau_{\text{cut}} | b, a, T) \), the joint probability density of the positions reached at time \( \tau_{\text{cut}} \), given that the paths start in \( a \) at \( \tau = 0 \) and end in \( b \) at \( \tau = T \). The answer to this question is well known from quantum mechanics: Cut the process in two time intervals, one from \( \tau = 0 \) to \( \tau = \tau_{\text{cut}} \) and the second piece from \( \tau = \tau_{\text{cut}} \) to \( \tau = T \). Because of the Markov property, these two parts are statistically independent, and one should take the product of propagators to obtain the joint PDF of the positions \( x_1(\tau_{\text{cut}}) \) and \( x_2(\tau_{\text{cut}}) \) at time \( \tau_{\text{cut}} \), given that the initial and final conditions are, respectively, \( (a, 0) \) and \( (b, T) \):

\[
Q_2(y, \tau_{\text{cut}} | b, a, T) = \frac{p_2(b, T | y, \tau_{\text{cut}}) p_2(y, \tau_{\text{cut}} | a, 0)}{Z_2(b, a, T)}, \tag{22}
\]
where \( Z_2(b,a,T) \) is the normalization constant, obtained by integrating over all intermediate points, 
\[
Z_2 = \int \text{d}y \, p_2(b,T|y,\tau_{\text{cut}}) p_2(y,\tau_{\text{cut}}|a,0),
\]
where we remember that the integration over \( y \) is ordered: 
\[
\int \text{d}y = \int_{-\infty}^{y_{\text{cut}}} \text{d}y_1 \int_{y_{\text{cut}}}^{\infty} \text{d}y_2.
\]
Using the closure relation of the states \( |y\rangle \), one finds that 
\[
Z_2(b,a,T) = p_2(b,T|a,0). \tag{24}
\]
Because we do not care about the lowest path \( x_1(\tau) \) and its intermediate position \( y_1 \) in the following, we compute
the marginal by integrating over all possible values of \( y_1 = x_1(\tau_{\text{cut}}) \), obtaining \( P_2(y_2,\tau_{\text{cut}}|b,a,T) \), the probability density
that the upper path is in \( y_2 \) at time \( \tau_{\text{cut}} \) given that the paths start in \( a \) at time \( \tau = 0 \) and end in \( b \) at time \( \tau = T \):
\[
P_2(y_2,\tau_{\text{cut}}|b,a,T) = \int_{-\infty}^{y_2} \text{d}y_1 \, Q_2(y,y_{\text{cut}}|b,a,T)
= \frac{\int_{-\infty}^{y_2} \text{d}y_1 \, p_2(b,T|y,\tau_{\text{cut}}) p_2(y,\tau_{\text{cut}}|a,0)}{p_2(b,T|a,0)}. \tag{25}
\]
From now on we set, for simplicity, \( T = 1 \). This can be
done without loss of generality because the Brownian scaling implies \( M \propto T^{1/2} \) and \( \tau_M \propto T \). A simple dimensional analysis makes it possible to reinsert the time \( T \) in our calculations.

### D. Maximum and the regularization procedure

The maximum \( M \) and the time to reach it \( \tau_M \) are defined by
\[
M = \max_{0 < \tau < 1} x_2(\tau) = x_2(\tau_M). \tag{26}
\]
Hence, to compute the joint probability of the couple \((M,\tau_M)\),
we want to impose \( y_2 = M \) and \( \tau_{\text{cut}} = \tau_M \), knowing that the
upper path does not cross the line \( x = M \). This condition
is implemented by inserting the product of Heaviside step functions
\[
\prod_{0 < \tau < 1} \theta[M - x_2(\tau)],
\]
in the path integral formula (14). The propagator of two vicious
Brownian paths, with the constraint that they do not cross \( x = M \), reads
\[
p_{<M,2}(b,t_b|a,t_a) = \frac{1}{Z_{<M}} \int_{x_1(t_a) = a_1}^{x_1(t_b) = b_1} \mathcal{D}x_1(\tau) \int_{x_2(t_a) = a_2}^{x_2(t_b) = b_2} \mathcal{D}x_2(\tau)
\times \left\{ \prod_{t_a < \tau < t_b} \theta[M - x_2(\tau)] \theta[M - x_1(\tau)] \theta[x_2(\tau) - x_1(\tau)] e^{-\int_{t_a}^{t_b} \frac{1}{2} k_0 \phi_k(x)^2 \text{d}t} e^{-\int_{t_a}^{t_b} \frac{1}{2} k_0 \phi_k(x)^2 \text{d}t} \right\} \tag{28}
\]
with the associated energies
\[
E_{k_j} = \frac{k_j^2}{2}, \quad k_j > 0. \tag{29e}
\]
Hence, with this information, one is able to compute the propagator \( p_{<M,2}(b,t_b|a,t_a) \) using the spectral decomposition,
as in Eq. (19):
\[
p_{<M,2}(b,t_b|a,t_a) = \langle b | e^{-(t_b-t_a)H_{-M}} | a \rangle
= \int_0^{\infty} \text{d}k_1 \int_0^{\infty} \text{d}k_2 \Phi_k(b)\Phi_k^*(a) e^{-(t_b-t_a)k^2} \pi, \tag{29f}
\]
with \( \Phi_k(x) \) the Slater determinant built with the one-particle
eigenfunctions (29d).

To obtain the joint PDF of \( M \) and \( \tau_M \), one would insert this
propagator in formulas (22)–(25), with \( y_2 = M \) and \( \tau_{\text{cut}} = \tau_M \).
With this procedure one naively finds zero, because we impose
the intermediate point \( y_2 \) to be on the edge of the hard wall
potential. This problem originates in the continuous nature of
the BM: Once in a point, the BM explores its vicinity immediately before and after [74]. This implies that we cannot
prescribe the upper path to be in \( x_2(\tau_M) = M \) without being
in \( x_2(\tau) > M \) for \( \tau \) close to \( \tau_M \). Another problem is that
the normalization constant cannot be written as a single
propagator. Indeed, one cannot insert the closure relation as we

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For the full context, please refer to the original source: [JOACHIM RAMBEAU AND GRÉGOIRE SCHEHR PHYSICAL REVIEW E 83, 061146 (2011)](http://journals.aps.org/pre/abstract/10.1103/PhysRevE.83.061146)
did from Eq. (23) to Eq. (24), because the potential depends actually on the position $y_2 \equiv M$.

Hence, one should prescribe a regularization scheme to avoid this phenomenon. A common way to deal with it is to take $y_2 = x_2(\tau_M) = M - \eta$, with $\eta > 0$ a small regularization parameter, compute all quantities as functions of $\eta$, and at the end take the limit $\eta \to 0$. This regularization scheme is similar to that introduced in Ref. [19]. Following the same ideas, one obtains the joint PDF as

$$P_2(M, \tau_M | \mathbf{b}, \mathbf{a}) = \lim_{\eta \to 0} \frac{W_2(M - \eta, \tau_M | \mathbf{b}, \mathbf{a})}{Z_2(\eta | \mathbf{b}, \mathbf{a})},$$

(30a)

where $W_2(M - \eta, \tau_M | \mathbf{b}, \mathbf{a})$ is the probability weight of all paths starting in $\mathbf{a}$ and ending in $\mathbf{b}$ in the unit time interval and such that $x_2(\tau_M) = M - \eta$, and $Z_2(\eta | \mathbf{b}, \mathbf{a})$ is the normalization constant. Explicitly, one has

$$W_2(M - \eta, \tau_M | \mathbf{b}, \mathbf{a}) = \int_{-\infty}^{\infty} \! dy_2 \int_{-\infty}^{\tau_M} \! dy_1 \left[ \delta[y_2 - (M - \eta)] \right. \times \left. p_{<M,2}(\mathbf{b}, y, \tau_M) p_{<M,2}(y, \tau_M | \mathbf{a}, 0) \right]$$

(30b)

for all $M > \max(a_2, b_2)$ and where we write a dummy integration over $y_2$ with the $\delta$ function forcing $y_2$ to be $M - \eta$. The normalization depends on the regularization parameter $\eta$:

$$Z_2(\eta | \mathbf{b}, \mathbf{a}) = \int_0^1 \! d\tau_M \int_{\max(a_2, b_2)}^\infty \! dM \; W_2(M - \eta, \tau_M | \mathbf{b}, \mathbf{a}).$$

(30c)

In the following paragraphs, all three configurations have their paths beginning at the origin. The problem of the continuous Brownian paths forbids to take directly $\mathbf{a} = (0, 0)$. Instead of that, we separate artificially the paths by an amount $\epsilon > 0$. Details of this regularization are left in the concerned paragraphs.

### E. Periodic boundary condition: $N = 2$ vicious Brownian bridges

In this case, the two paths start from the origin at $\tau = 0$ and arrive also at the origin at final time $\tau = 1$: $x_1(0) = x_2(0) = x_1(1) = x_2(1) = 0$. Without extra condition, this defines two Brownian bridges. Here we compute the joint probability density function for two Brownian bridges, under the condition that they do not cross each other (see Fig. 4).

As discussed before, there is again a regularization scheme to adopt in both starting and ending points. We separate by an amount $\epsilon$ the two paths at the starting and ending points, taking $\mathbf{a} = \mathbf{b} = (\epsilon, \epsilon) = \mathbf{\epsilon}$. At the end of the computation, we take the limit $\epsilon \to 0$. Then the joint probability density function for two vicious Brownian bridges is

$$P_{2,\mathbf{\epsilon}}(M, \tau_M) = \lim_{\epsilon \to 0} P_2(M, \tau_M | \mathbf{\epsilon}, \mathbf{\epsilon}),$$

(31a)

where $P_2(M, \tau_M | \mathbf{b}, \mathbf{b})$ is defined in Eq. (30a). The numerator of Eq. (30a), the probability weight, reads

$$W_{2,\mathbf{\epsilon}}(M - \eta, \tau_M | \mathbf{\epsilon}, \mathbf{\epsilon}) = W_2(M - \eta, \tau_M | \mathbf{\epsilon}, \mathbf{\epsilon}),$$

(31b)

and the denominator of Eq. (30a), that is, the normalization, is given by

$$Z_{2,\mathbf{\epsilon}}(\eta | \mathbf{\epsilon}, \mathbf{\epsilon}) = Z_2(\eta | \mathbf{\epsilon}, \mathbf{\epsilon}),$$

(31c)

where the subscript “$\mathbf{B}$” refers to “Bridges,” and with the relation deduced from Eq. (30a)

$$P_{2,\mathbf{\epsilon}}(M, \tau_M) = \lim_{\eta \to 0, \epsilon \to 0} \frac{W_{2,\mathbf{\epsilon}}(M - \eta, \tau_M | \mathbf{\epsilon}, \mathbf{\epsilon})}{Z_{2,\mathbf{\epsilon}}(\eta | \mathbf{\epsilon}, \mathbf{\epsilon})}.$$
The same operations apply to $\Phi_k(y)$:
\[
W_{2,B}(M - \eta, \tau_M, \epsilon) = \eta^2 \sqrt{\frac{2}{\pi M}} \int_{-\infty}^{\infty} dy_1 \int_{0}^{\infty} dk' k_1' \Phi_k(y_1) \Phi_k^*(\epsilon) e^{-(1-\tau_M)k_1'^2 + \epsilon} \times \int_{0}^{\infty} dk k_1 \phi_{\epsilon,k_1}(y_1) \Phi_k^*(\epsilon) e^{-\tau_M k_1^2}. 
\]
(35)

Permuting the order of integrations, one can compute the integration with respect to $y_1$ as
\[
\int_{-\infty}^{\infty} dy_1 \Phi_k^*(y_1) \phi_{\epsilon,k_1}(y_1) = \delta(k_1 - k_1') + O(\eta), 
\]
(36)
due to the orthonormalization of the eigenfunctions $\phi_{\epsilon}(x)$.

Performing the integration over $k_1'$ leads to, at lowest order in $\eta$,
\[
W_{2,B}(M - \eta, \tau_M, \epsilon) = \frac{4}{\pi} \eta^2 \int_{0}^{\infty} dk_1 e^{-\frac{\epsilon}{2}} \int_{0}^{\infty} dk_2 \int_{0}^{\infty} dk_2' \times \left\{ k_1 k_2 \Phi_{k_1,k_2}(\epsilon) \Phi_{k_1,k_2}(\epsilon) e^{-(1-\tau_M)k_1^2 + \epsilon} + O(\eta^3) \right\}, 
\]
(37)
The next step is to expand in powers of $\epsilon$ inside the Slater determinants: The first column does not depend on $\epsilon$, and in the second column, we expand at order $\epsilon$, each of the two elements, swapping the constant term by linear combination with the first column. This yields
\[
\Phi_{k_1,k_2}(\epsilon) = \sqrt{\frac{1}{2}} \Phi_k(0) \Phi_k(\epsilon) = \frac{\epsilon}{\sqrt{2}} \left| \Phi_k(0) \Phi_k'(0) \right| + O(\epsilon^3) 
\]
where we use the scaled variables $q_i = k_i M$ and the determinant
\[
\Theta_2(q_1,q_2) = \det_{1 \leq i,j \leq 2} \left[ q_i^{-1} \cos(q_j - j \frac{\pi}{2}) \right]. 
\]
(39)

Performing the same expansion in powers of $\epsilon$ in $\Phi_{k_1,k_2}(\epsilon)$, with $q'_i = k'_i M$, one can identify the dominant term as a factor of the product of powers $\eta^3 \epsilon^2$, so that
\[
W_{2,B}(M - \eta, \tau_M, \epsilon) = \eta^2 \epsilon^2 \epsilon W_{2,B}(M, \tau_M) + O(\eta^3 \epsilon^2, \eta^2 \epsilon^3). 
\]
(40)

Inserting the expansion (38) in (37), and identifying the leading term, one obtains
\[
W_{2,B}(M, \tau_M) = \frac{8}{\pi^3 M^2} \int_{0}^{\infty} dq_1 e^{-\frac{q_1^2}{2M^2}} \times \left\{ \int_{0}^{\infty} dq_2 q_2' \Theta_2(q_1,q_2) e^{-(1-\tau_M)q_2'^2 \frac{M^2}{2}} \times \int_{0}^{\infty} dq_2' \Theta_2(q_1,q_2') e^{-(1-\tau_M)q_2'^2 \frac{M^2}{2}} \right\}. 
\]
(41)

It can be written in the more compact form
\[
W_{2,B}(M, \tau_M) = 8 \sqrt{\frac{\pi}{M^2}} \int_{0}^{\infty} dq_1 e^{-\frac{q_1^2}{2M^2}} \times \gamma_2(q_1 | M, 1 - \tau_M) \gamma_2(q_1 | M, \tau_M), 
\]
(42)
with the help of the function $\gamma_2$,
\[
\gamma_2(q_1 | M, \tau_M) = \sqrt{\frac{M}{\sqrt{2 \tau_M}}} e^{-\frac{M^2}{2 \tau_M}} \cos \left( q_1 - \frac{\pi}{2} \right) \cdot \left| q_1 \cos(q_1 - \pi) \right| \times 
\]
(43)
\[
H_1 \left( \frac{M}{\sqrt{2 \tau_M}} \right) \left( \frac{M}{\sqrt{2 \tau_M}} \right) H_2 \left( \frac{M}{\sqrt{2 \tau_M}} \right). 
\]
(44)

$H_1(\chi)$ and $H_2(\chi)$ are the first and second Hermite polynomials. This expression is obtained by factorizing all the $q_2$ dependence in the last line of the determinant and performing the integration directly in this last line. The Hermite polynomials then appear naturally (see Appendix E). Then one proceeds to the expansion of the determinants $\gamma_2$ with respect to their last lines, which permits the integration over $q_1$ (four terms, because there are two determinants $2 \times 2$), and one obtains a simple, though long, expression (apart from the exponentials, only algebraic terms appear).

The limit in Eq. (31d) exists provided the normalization constant admits the following expansion:
\[
Z_{2,B}(\eta, \epsilon) = \eta^2 \epsilon^2 K_{2,B} + o(\eta^3 \epsilon^3), 
\]
where $K_{2,B}$ is a number (independent of $M$ and $\tau_M$). It can be computed using the normalization of the joint PDF:
\[
1 = \int_{0}^{\infty} dM \int_{0}^{\infty} d\tau_M P_{2,B}(M, \tau_M) = \frac{1}{K_{2,B}} \int_{0}^{\infty} dM \int_{0}^{\infty} d\tau_M W_{2,B}(M, \tau_M). 
\]
(46)

One finds $K_{2,B} = 1/\pi$. The joint PDF reads
\[
P_{2,B}(M, \tau_M) = \sqrt{\frac{\pi}{\tau_M(1 - \tau_M)^{3/2}}} e^{-\frac{M^2}{2(1 - \tau_M)}} \times 
\]
(47)
\[
\frac{1}{4} H_2 \left( \frac{M}{\sqrt{2 \tau_M}} \right) H_2 \left( \frac{M}{\sqrt{2(1 - \tau_M)}} \right) \times \left( 1 - e^{-2M^2} \right) + M^2 e^{-2M^2} \times 
\]
\[
\left[ H_2 \left( \frac{M}{\sqrt{2 \tau_M}} \right) H_2 \left( \frac{M}{\sqrt{2(1 - \tau_M)}} \right) \right] \times \left[ 1 - \frac{1}{2} e^{-2M^2} H_2 \left( \sqrt{2M} \right) \right]. 
\]

where for compactness we use the Hermite polynomial $H_2(\chi) = 4\chi^2 - 2$. A direct integration over $M$ gives the
FIG. 5. (Color online) Distribution of the time at which the maximum is reached for $N = 2$ nonintersecting vicious Brownian paths. The solid line corresponds to our analytical formula in Eq. (48), while the symbols correspond to the results of our numerical simulations.

The probability distribution function of $\tau_M$, the time at which the maximum is reached, regardless to the value of the maximum (see Fig. 5):

$$P_{2,B}(\tau_M) = 4 \left(1 - \frac{1 + 10\tau_M(1 - \tau_M)}{1 + 4\tau_M(1 - \tau_M)}\right)^{5/2}. \quad (48)$$

The fact that $\tau_M$ enters in this expression only through the product $\tau_M(1 - \tau_M)$ reflects the symmetry of the distribution around $\tau_M = 1/2$, which of course is expected for periodic boundary conditions. To center the distribution, over a unit length interval, one can use $\tau_M = 1/2 + u_M/2$, and the distribution of $u_M$ is then

$$P_{2,B}^{\text{centered}}(u_M) = 2 - \frac{5}{(2 - u_M^2)^{3/2}} + \frac{3}{(2 - u_M^2)^{5/2}}. \quad (49)$$

Two asymptotic analysis can be made:

(i) for $\tau_M \simeq 0$ (or equivalently $\tau_M \simeq 1$), one has

$$P_{2,B}(\tau_M) \sim 120\tau_M^2 + O(\tau_M^3);$$

(ii) for $\tau_M$ in the vicinity of $1/2$, better written in terms of the behavior in $u_M = 0$ of the centered distribution,

$$P_{2,B}^{\text{centered}}(u_M) \sim \left(4 - \frac{7}{2\sqrt{2}}\right) - \frac{15u_M^2}{8\sqrt{2}} + O(u_M^3). \quad (50)$$

Furthermore, an integration of $P_{2,B}(M, \tau_M)$ with respect to $\tau_M$ gives the PDF of the maximum $F_{2,B}(M)$ (the derivative of the cumulative distribution). Using the change of variables $\tau_M = [1 + \sin(\varphi)]/2$, the integral gives

$$F_{2,B}(M) = \int_0^1 d\tau_M P_{2,B}(M, \tau_M) = 8Me^{-4M^2} - 8Me^{-2M^2} + 16M^3e^{-2M^2},$$

which coincides with the result computed in Ref. [30].

F. Free boundary conditions: $N = 2$ stars configuration

In this section, we consider two vicious Brownian paths starting from the origin with free end points: This is the star configuration (a subscript “S” is used; see Fig. 6). The starting points are the same as before $a = \epsilon = (0, \epsilon)$, but the end points are free, so that for $M > 0$ fixed, we sum over all end points $b$ such that $b_1 < b_2 < M$. Hence, one has

$$P_{2,S}(M, \tau_M) = \lim_{\epsilon \to 0^-} \frac{W_{2,S}(M - \eta, \tau_M, \epsilon)}{Z_{2,S}(\eta, \epsilon)}. \quad (51)$$

where the probability weight is given by

$$W_{2,S}(M - \eta, \tau_M, \epsilon) = \int_{\text{ord}} db \int_{\text{ord}} dy \left[\delta[y_2 - (M - \eta)] - \left[p_{<,M,2}(b,1|y,\tau_M) - p_{<,M,2}(b,2|y,\tau_M|\epsilon,0)\right]\right], \quad (52)$$

where the integrations satisfy $-\infty < b_1 < b_2 < M$ and $-\infty < y_1 < y_2 < M - \eta$, respectively. The normalization is such that

$$Z_{2,S}(\eta, \epsilon) = \int_0^1 d\tau_M \int_0^M dm W_{2,S}(M - \eta, \tau_M, \epsilon). \quad (53)$$

As in the previous section, we use only the leading order term of the expansion of $W_{2,S}$ and $Z_{2,S}$ in powers of $\eta$ and $\epsilon$. The computation of the probability weight can be done along the same lines as before (33)–(37), keeping in mind that only the final points change. At lowest order in $\eta$, one has

$$W_{2,S}(M - \eta, \tau_M, \epsilon) = \int_{-\infty}^{M} db_1 \int_{-\infty}^{b_2} db_2 \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk',$$

$$\times e^{-\frac{\eta^2}{4} + \frac{\eta^2}{2} + k^2 - \frac{\eta}{2} - \frac{\eta}{2} + \delta(k', k|\epsilon)} e^{-(1 - \mu)^2/4 - \eta\epsilon/2}, \quad (54)$$

where we have integrated over $y_1$ first, and then over $k'_1$, with the $\delta$ function coming from the closure relation of eigenfunctions. For the eigenfunctions evaluated in the final points one obtains

$$\Phi_{k_1,k_2}(b) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_{k_1}(b_1) & \phi_{k_2}(b_2) \\ \phi_{k_1}(b_2) & \phi_{k_2}(b_1) \end{pmatrix} = \frac{\sqrt{2}}{\pi} \left[\sin(q_1 \xi_1) \sin(q_2 \xi_2) - \sin(q_2 \xi_1) \sin(q_1 \xi_2)\right].$$

FIG. 6. (Color online) One realization of $N = 2$ nonintersecting free BMs, that is, in “star” configuration, with the maximum $M$ and the time $\tau_M$ at which this maximum is reached.
in terms of the previous $q_j = k_j M$, and with the variables
\[ \zeta_j = \frac{M - k_j M}{\mathcal{M}} \] (for $j = 1, 2$). Using the expansion in $\varepsilon$ written in Eq. (38), one finds the leading order to be $O(\varepsilon^2)$ so that the expansion is
\[
W_{2,S}(M - \eta, \tau_M, \varepsilon) = \eta^2 \varepsilon W_{2,S}(M, \tau_M) + O(\varepsilon^3, \eta^2 \varepsilon^2),
\]
with
\[
W_{2,S}(M, \tau_M) = \frac{8}{\pi^3 M^4} \int_0^\infty d\zeta_2 \int_{\zeta_2}^\infty d\zeta_1 \int_0^\infty d\zeta_2' \int_{\zeta_2'}^\infty d\zeta_1' \times \int_0^\infty dq_1 \int_0^\infty dq_2 \Theta_2(q_1, q_2) e^{-\frac{q_2^2}{2M^2} - \frac{1}{\varepsilon} q_1^2 + \frac{1}{\varepsilon} q_2^2 + \frac{1}{\varepsilon} q_1^2 \zeta_2^2} \times [\sin(q_1 \zeta_1) \sin(q_2 \zeta_2') - \sin(q_2 \zeta_2) \sin(q_1 \zeta_1')],
\]
where $\Theta_2(q_1, q_2)$ is given in Eq. (39). As before, the integration over the moments $q_2$ and $q_2'$, corresponding to the top walker, can be factorized in the determinants. From the part $[0, \tau_M]$, one recognizes $\Upsilon_2(q_1|M, \tau_M)$, and for the part $[\tau_M, 1]$, we introduce
\[
\tilde{\Upsilon}_2(q_1, \zeta|1 - \tau_M) = \int_0^\infty dq_2 q_2^2 e^{-\frac{1}{\varepsilon} q_2^2 + \frac{1}{\varepsilon} q_2^2 \zeta_1^2} \times [\sin(q_1 \zeta_1) \sin(q_2 \zeta_2') - \sin(q_2 \zeta_2) \sin(q_1 \zeta_1')],
\]
With the help of the two functions $\Upsilon_2$ and $\tilde{\Upsilon}_2$, one obtains
\[
W_{2,S}(M, \tau_M) = \frac{8}{\pi^3 M^4} \int_0^\infty d\zeta_2 \int_{\zeta_2}^\infty d\zeta_1 \int_0^\infty d\zeta_2' \int_{\zeta_2'}^\infty d\zeta_1' \times \int_0^\infty dq_1 \int_0^\infty dq_2 \Upsilon_2(q_1|M, \tau_M) \tilde{\Upsilon}_2(q_1, \zeta|M, 1 - \tau_M).
\]
$\Upsilon_2$ and $\tilde{\Upsilon}_2$ can be written as determinants, as in Eq. (44), and
\[
\tilde{\Upsilon}_2(q_1, \zeta|1 - \tau_M) = \sqrt{\pi} \left( \frac{M}{\varepsilon^2} \right)^2 \times \left[ \sin(q_1 \zeta_1) e^{-\frac{q_2^2}{2M^2} \zeta_1^2} H_1 \left( \frac{M}{\varepsilon^2} \zeta_1 \right) e^{-\frac{q_2^2}{2M^2} \zeta_1^2} H_1 \left( \frac{M}{\varepsilon^2} \zeta_2' \right) \right],
\]
where we used the relation in Eq. (E3), and where $t$ stands for $1 - \tau_M$ (for compactness). The fact that we do not have the periodic boundary conditions introduces an asymmetry; thus, the two determinants $\Upsilon_2$ and $\tilde{\Upsilon}_2$ do not have the same form.

As before, we also expand the normalization $Z_{2,S}(\eta, \varepsilon)$ in powers of $\eta$ and $\varepsilon$, the leading term being
\[
Z_{2,S}(\eta, \varepsilon) = \eta^2 \varepsilon K_{2,S} + O(\eta^3 \varepsilon^2),
\]
with $K_{2,S}$ a number, independent of $M$ and $\tau_M$, which can be computed by the normalization condition:
\[
1 = \int_0^\infty dM \int_0^1 d\tau_M \mathcal{P}_{2,S}(M, \tau_M)
= \frac{1}{K_{2,S}} \int_0^\infty dM \int_0^1 d\tau_M W_{2,S}(M, \tau_M).
\]
which can be compared to the divergence of the distribution of the time \( \tau_M \) for one BM (the Lévy arcsine law, plotted in Fig. 7) \( P_{1,2}(\tau_M) = \frac{1}{\pi} [\tau_M (1 - \tau_M)]^{-1/2} \). The fact that we have another uncrossing Brownian path below preserves the exponent of the divergence, but changes slightly the prefactor.

Notice finally the mean value is given by \( \langle \tau_M \rangle_{2.S} = 7/2 - 2 \sqrt{2} = 0.671573 \ldots \) and that the sign of the derivative of \( P_{2,3}(\tau_M) \) changes in \( \tau_M \approx 0.451175 \).

G. Periodic boundary condition and positivity constraint: \( N = 2 \) excursions configuration

Now we come to the computation of the joint probability distribution of the maximum \( M \) and its position \( \tau_M \) for the configuration of two vicious Brownian paths subjected to stay in the half line \( x > 0 \), in addition to the periodic condition: the two paths start in \( x = 0 \) and end in \( x = 0 \) at \( \tau = 1 \) (see Fig. 8). This corresponds to two Brownian excursions, with the noncrossing condition (we use the subscript " \( \Box \)"").

The positivity constraint is easily implemented in the path integral formulation by adding another hard wall in \( x = 0 \). The potential associated with each of the two fermions then reads

\[
V_\Box(x) = \begin{cases} 0 & \text{if } 0 < x < M, \\ +\infty & \text{elsewhere}. \end{cases}
\]

The eigenfunctions of the one-particle Hamiltonian

\[
H_\Box^{(j)}(x_j) = -\frac{1}{2} \frac{d^2}{dx_j^2} + V_\Box(x_j)
\]

are given by

\[
\psi_n(x) = \sqrt{\frac{2}{M}} \sin \left( \frac{n\pi}{M} x \right), \quad n = 1, 2, \ldots,
\]

with the associated eigenvalues

\[
E_n = \frac{n^2 \pi^2}{2M^2}.
\]

This makes it possible to express the corresponding propagator, for any point \( a, b \) and time \( 0 \leq \tau_a < \tau_b \leq 1 \), as

\[
p_{\Box}(b, \tau_b|a, \tau_a) = \langle b | e^{-(\tau_b - \tau_a)H_\Box} | a \rangle
\]

\[
= \sum_{n_1, n_2 > 0} \psi_{n_1}(a) \psi_{n_2}^*(b) e^{-(\tau_b - \tau_a)\frac{n_1^2}{2M^2} - \frac{n_2^2}{2M^2}},
\]

(63e)

where \( \mathbf{n} = (n_1, n_2) \) is a couple of two positive integers, and in which we use the Slater determinant \( \psi_\Box(a) = \det_{1 \leq i, j \leq 2} [\psi_n(a_j)]/\sqrt{2!} \).

Here the regularization is necessary for the boundary values of both fermions, because the lowest path cannot start and end exactly in \( x = 0 \). Therefore, we put \( x(0) = a = (\epsilon, 2\epsilon) = \epsilon \) and \( x(1) = b = \epsilon \), where the notation \( \epsilon = (\epsilon, 2\epsilon) \) differs slightly from the previous one \( \epsilon = (0, \epsilon) \). For this geometry, one uses Eqs. (30a)–(30c) to obtain the probability weight

\[
W_{2,E}(M - \eta, \tau_M, \epsilon) \equiv W_2(M - \eta, \tau_M; \epsilon, \epsilon)
\]

\[
= \int_{\text{ord}} dy [\delta[y_2 - (M - \eta)]]
\times p_{\Box}(\epsilon, l | y, \tau_M) p_{\Box}(y, \tau_M | \epsilon, 0),
\]

(64a)

where the ordered integral covers the domain \( 0 < y_1 < y_2 < M \). The normalization function is

\[
Z_{2,E}(\eta, \epsilon) = Z_2(\eta; \epsilon, \epsilon),
\]

(64b)

so that

\[
P_{2,E}(M, \tau_M) = \lim_{\epsilon \to 0, \eta \to 0} \frac{W_{2,E}(M - \eta, \tau_M, \epsilon)}{Z_{2,E}(\eta, \epsilon)}.
\]

(64c)

Inserting the spectral decomposition of the propagator (63e) in Eq. (64a), one has

\[
W_{2,E}(M - \eta, \tau_M, \epsilon) = \int_{\text{ord}} dy \delta[y_2 - (M - \eta)]
\times \sum_{n, n' > 0} \left[ \psi_{n}(\epsilon) \psi_{n'}^*(\epsilon) e^{-\frac{n^2 \epsilon^2}{2M^2}} \psi_{n}(y) \psi_{n'}^*(\epsilon) e^{-\frac{n^2 \epsilon^2}{2M^2}} \right].
\]

(65)

We follow the same procedure as before in Eqs. (33)–(37): Using the symmetry in the exchange of indices \( n_1 \leftrightarrow n_2 \) and \( n'_1 \leftrightarrow n'_2 \) we express the two Slater determinants as the product of their diagonal components:

\[
W_{2,E}(M - \eta, \tau_M, \epsilon) = \sum_{n, n'} \left[ \psi_{n}(\epsilon) \psi_{n'}^*(\epsilon) e^{-\frac{\epsilon^2}{2M^2}} \psi_{n}(y) \psi_{n'}^*(\epsilon) e^{-\frac{n^2 \epsilon^2}{2M^2}} \right]
\times \psi_{n_1}(M - \eta) \psi_{n_2}(M - \eta) \int_0^{M - \eta} dy_1 \psi_{n_1}^*(y_1) \psi_{n_1}(y_1).
\]

(66)
An expansion at lowest order in $\eta$ of this expression, making use of

$$\psi_n(M - \eta) = (-1)^{n+1} \sqrt{\frac{2 M}{M}} \eta + O(\eta^3),$$  \tag{67}


together with the closure relation of the eigenfunctions

$$\int_0^M dy_1 \psi_{n_1}(y_1) \psi_{n_1'}(y_1) = \delta_{n_1,n_1'},$$  \tag{68}


yields

$$W_{2,E}(M - \eta, \tau_M, \epsilon) = \eta^2 \epsilon \frac{2 \pi^2}{M^3} \sum_{n,n'} \left\{ \psi_n(\epsilon) \psi_{n'}(\epsilon) \times e^{\frac{-\pi^2}{2 \tau_M^2} (\eta_{n_1,n_1'-\tau_M n'_{1}} - \eta_{n_1,n_1'} - 1)} - e^{\frac{-\pi^2}{2 \tau_M^2} (\eta_{n_1,n_1'-\tau_M n'_{1}} - \eta_{n_1,n_1'} + 1)} \right\}. \tag{69}

$$

The expansion to the lowest order in $\epsilon$ of the eigenfunctions reads

$$\psi_n(\epsilon) = (-1)\frac{4\sqrt{2\pi}^4}{3M^3} \eta n_1 n_2 \Delta_2(n_1^2,n_2^2) \epsilon^4 + O(\epsilon^6). \tag{70}$$

where $\Delta_2(\lambda_1,\lambda_2)$ is the $2 \times 2$ Vandermonde determinant

$$\Delta_2(\lambda_1,\lambda_2) = \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = (\lambda_2 - \lambda_1).$$

Hence, to lowest order, one has

$$W_{2,E}(M - \eta, \tau_M, \epsilon) \approx \frac{64\pi^{10}}{9M^{13}} \sum_{n_1 \geq 0} \left\{ n_1^2 \epsilon - \frac{\pi^2}{6 \pi^2} \right\} \times \sum_{n_2 \geq 0} (-1)^{n_1} n_2^2 \Delta_2(n_1^2,n_2^2) e^{-\frac{\pi^2}{2 \tau_M^2} \tau_M n_2^2} \times \sum_{n_3 \geq 0} (-1)^{n_1} n_3^2 \Delta_2(n_1^2,n_3^2) e^{-\frac{\pi^2}{2 \tau_M^2} (1-\tau_M n_3^2)}. \tag{71}$$

To ensure the joint PDF to exist, the normalization must have the same dominant term

$$Z_{2,E}(\eta,\epsilon) = \eta^2 e^{\frac{\pi^2}{2 \tau_M^2}} K_{2,E} + O(\eta^2 e^{\frac{\pi^2}{2 \tau_M^2}}).$$

Combining these expansions in Eq. (71) and in Eq. (72) one obtains from Eq. (64c)

$$P_{2,E}(M, \tau_M) = \frac{\pi^{11}}{3M} \sum_{n_1,n_2,n_3 \geq 0} (-1)^{n_2+n_3} n_1^2 n_2^2 n_3^2 \times (n_2^2 - n_3^2)(n_3^2 - n_1^2) e^{-\frac{\pi^2}{2 \tau_M^2} (1-\tau_M n_1^2 + \tau_M n_2^2 + \tau_M n_3^2)}, \tag{72}$$

where the computation of normalization has been left in Appendix C.

This formula is well suited for an integration over $M$ to deduce the marginal law of $\tau_M$:

$$P_{2,E}(\tau_M) = \int_0^\infty dM P_{2,E}(M, \tau_M) \approx \frac{27}{\pi} \frac{1}{\tau_M} \int_0^\infty dx^5 \times \sum_{n_1,n_2,n_3} \left\{ (-1)^{n_2+n_3} n_1^2 n_2^2 n_3^2 (n_2^2 - n_3^2)(n_3^2 - n_1^2) \right\} e^{-\frac{\pi^2}{2 \tau_M^2} (1-\tau_M n_1^2 + \tau_M n_2^2 + \tau_M n_3^2)}. \tag{75}$$

When $\tau_M \rightarrow 0$, the argument of the first exponential becomes $-\chi(n_1^2 + n_2^2)/\sqrt{\tau_M}$ at leading order, and the dominant term in the sums over $n_1$ and $n_3$ is given by $(n_1 = 1, n_3 = 2)$ and $(n_1 = 2, n_3 = 1)$ because of the factor $(n_2^2 - n_3^2)$, which enforces $n_1 \neq n_3$. Hence, to leading order, one has

$$P_{2,E}(\tau_M) \simeq \frac{27}{\pi} \frac{1}{\tau_M} \int_0^\infty dx^5 \times \sum_{n_1,n_3} (-1)^{n_2} n_2^2 (2n_2^3 - 5) e^{-\frac{\pi^2}{2 \tau_M} \tau_M n_2^2}. \tag{76}$$

FIG. 9. (Color online) Distribution of the time at which the maximum is reached for $N = 2$ nonintersecting excursions (watermelon with a wall). The solid line corresponds to our analytic result while the symbols are the results of our numerical simulations.
By differentiating $N$ times the Jacobi identity
\begin{equation}
1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-nz} = 2 \left( \frac{\pi}{z} \right)^{1/2} \sum_{n=0}^{\infty} e^{-\pi^2(n + \frac{1}{2})^2/z} \tag{77}
\end{equation}
with respect to $z$, one has at leading order when $z \to 0$
\begin{equation}
\sum_{n=1}^{\infty} (-1)^n 2^n n^{2n-2} e^{-n^2 z} \sim \frac{\pi}{z} \left( \frac{\pi}{2} \right)^2 N \frac{1}{z} e^{-z^2/\pi}. \tag{78}
\end{equation}
With $z = x \sqrt{\tau_M}$, one obtains at leading order
\begin{equation}
P_{2,E}(\tau_M) \approx 2^4 \pi^{7/2} \tau_M^{(5 + 1)/2} \int_0^{\infty} dx \, x^{1/2} e^{-\left(5x^2 + \frac{z^2}{\tau_M}\right)} \frac{1}{\sqrt{\pi x}} \tag{79}
\end{equation}
which can be approximated by a saddle point method. At leading order when $\tau_M \to 0$, one obtains
\begin{equation}
P_{2,E}(\tau_M) \approx \frac{8\pi^5}{5} \frac{1}{\tau_M} e^{-\frac{z^2}{\tau_M}}. \tag{80}
\end{equation}
By symmetry one obtains the same behavior when $\tau_M \to 1$ by replacing $\tau_M$ by $1 - \tau_M$ in the above formula.

As in the previous paragraphs, it is possible to have a compact formula for the joint PDF, anticipating the general computation for $N$ vicious Brownian paths. Let us introduce $\Omega_2$, the analog of $\Omega_2$ (43) in this discrete case:
\begin{equation}
\Omega_2(n_1|M, t) = \sum_{k=0}^{\infty} (-1)^k k^2 \Delta_2(n_1,k^2) e^{-\frac{\pi^2}{\tau_M} tk^2}, \tag{81}
\end{equation}
with $k$ a generic index standing for $n_2$ ($t = \tau_M$) or $n_3$ ($t = 1 - \tau_M$). Factorizing the sum over $k$ in the last line of the Vandermonde determinant and using sums defined by
\begin{equation}
\omega_0(M, t) = \sum_{k=0}^{\infty} (-1)^k k^2 e^{-\frac{\pi^2}{\tau_M} k^2}, \tag{82}
\end{equation}
one is able to write
\begin{equation}
\Omega_2(n_1|M, t) = \begin{vmatrix} 1 & n_1^2 \\ \omega_0(M, t) & \omega_2(M, t) \end{vmatrix}. \tag{83}
\end{equation}
Hence, the joint PDF can be expressed in a factorized way, reminiscent of the formula for the watermelon case (42)
\begin{equation}
P_{2,E}(M, \tau_M) = \frac{\pi^{11}}{3M^{13}} \sum_{n_1} n_1^2 e^{-\frac{\pi^2}{\tau_M} n_1^2} \times \Omega_2(n_1|M, \tau_M) \Omega_2(n_1|M, 1 - \tau_M). \tag{84}
\end{equation}
In the following we extend these formulas, for bridges and excursions, to $N$ particles for generic $N$.

### III. DISTRIBUTION OF THE POSITION OF THE MAXIMUM FOR $N$ VICIOUS BROWNIAN PATHS

In this section, we extend the previous method to any number $N$ of vicious Brownian paths, ordered as $x_1(\tau) < x_2(\tau) < \cdots < x_N(\tau)$ for $0 < \tau < 1$ (again we take a unit time interval, without loss of generality). We want to compute the joint PDF of $(M, \tau_M)$, respectively the maximum and the time to reach it for different configurations, with
\begin{equation}
M = \max_{0 \leq \tau \leq 1} x_N(\tau) = x_N(\tau_M). \tag{85}
\end{equation}
In the same way as in the previous section, the noncolliding condition translates into treating indistinguishable fermions. They do not interact between each other but are all subjected to the same potential [which is one or two hard wall(s) potential in our computations]. The Hamiltonian of the system is
\begin{equation}
H = \sum_{i=1}^{N} \epsilon_i \otimes \cdots \otimes H^{(i)} \otimes \cdots \otimes \pi^{(N)'}, \tag{86}
\end{equation}
with identical one-particle Hamiltonians $H^{(i)}$, for $1 \leq i \leq N$. If $\psi_i(x_i)$ is the eigenfunction of $H^{(i)}$ associated with the energy $E_i$, one can write the eigenfunction of the $N$-particles problem as a Slater determinant,
\begin{equation}
\psi_E(x) = \frac{1}{\sqrt{N!}} \det_{i,j \leq N} [\psi_E(x_i)] \tag{87}
\end{equation}
with $E = \sum_i E_i$ and where we used the bold type for $N$ vectors: $x = (x_1, x_2, \ldots, x_N)$. Hence, the propagator reads as in formula (19), but with determinants of $N \times N$ matrices. The method to compute the joint probability density function is exactly the same as before, replacing 2-vectors with $N$-vectors in formulas (30).

#### A. $N$ vicious Brownian bridges

We first consider the watermelon configuration, that is, the case of $N$ nonintersecting Brownian bridges (see definition in Sec. II E). The periodic boundary conditions impose the starting and ending points to be at the origin. As discussed above, this needs a regularization scheme, and we take $a = b = [0, \epsilon, 2\epsilon, \ldots, (N - 1)\epsilon] = \epsilon$. The joint PDF is
\begin{equation}
P_{N,\epsilon}(M, \tau_M) = \lim_{\epsilon \to 0} \frac{W_{N,\epsilon}(M - \eta, \tau_M, \epsilon)}{Z_{N,\epsilon}(\eta, \epsilon)}, \tag{88}
\end{equation}
where $W_{N,\epsilon}(M - \eta, \tau_M, \epsilon)$ is the probability weight of all paths starting and ending in $\epsilon$ with $x_N(\tau_M) = M - \eta$, given by
\begin{equation}
W_{N,\epsilon}(M - \eta, \tau_M, \epsilon) = \int_0^\infty d\eta [\delta(y_N - (M - \eta)) \times p_{-M,N}(\epsilon, 0 | y, \tau_M) p_{-M,N}(y, \tau_M | \epsilon, 0)]. \tag{89}
\end{equation}
$p_{-M,N}$ is the propagator of $N$ indistinguishable fermions, each of them having the one-particle Hamiltonian $H_{<M}$ (29b), with eigenfunctions $\phi_k(x)$ and eigenvalues $E_k$ given by Eqs. (29d) and (29e).

The starting point of our computation is given by Eq. (33) with the upper point now $y_N$. All the manipulations done before for 2 paths extend to $N$ paths. A subtlety is the fact that the ordered integral over the domain $-\infty < y_1 < y_2 < \cdots < y_{N-1} < y_N < M$ is proportional to the same unordered integral over the domain $-\infty < y_i < M$ for $1 \leq i \leq N$, because of the symmetry of the integrand. This is shown in Appendix A. With the same argument of symmetry between the exchange of any $k_i \leftrightarrow k_j$, and the same for $k'$ one could use only the diagonal terms for the two Slater determinants expressed in $y$. After permuting the $y$ integrals with the $k$ integrals, this operation gives $N - 1$ closure relations, that is, a product of $N - 1$ Dirac $\delta$ functions $\prod_{i=1}^{N-1} \delta(k_i - k'_i)$, and the product $\phi_{k_{M-N}}(M - \eta) \phi_{k_{M-N}}(M - \eta)$. The expansion of the Slater
Therefore, one obtains from Eqs. (91) and (92), together with Eq. (88),

\[ \Theta_N(q) = \det_{j=1}^{j=N} \left[ q_j^{j-1} \cos \left( q_j - j\frac{\pi}{2} \right) \right], \quad (90) \]

with the rescaled variables \( q_j = k_j M \). After the aforementioned manipulations, one finds at lowest order

\[ W_{N,B}(M - \eta, \tau_M, \epsilon) = \eta^2 e^{N(N-1)} W_{N,B}(M, \tau_M) + o(\eta^2 e^{N(N-1)}), \quad (91) \]

together with

\[ Z_{N,B}(\eta, \epsilon) = \eta^2 e^{N(N-1)} K_N, B + o(\eta^2 e^{N(N-1)}). \quad (92) \]

Therefore, one obtains from Eqs. (91) and (92), together with Eq. (88),

\[ P_{N,B}(M, \tau_M) = \frac{A_{N,B}}{M^{N+\frac{3}{2}}} \int_0^\infty dq_1, \ldots, dq_{N-1} e^{-\frac{\sum_{i=1}^{N-1} q_i^2}{2M}} \times \Theta_N(|q_j| |M, \tau_M) \Theta_N(|q_j| |M, 1 - \tau_M), \quad (93) \]

where

\[ \Theta_N(|q_j| |M, t) = \int_0^\infty dq_N q_N e^{-\frac{q_N^2}{2M} \Theta_N(q)} \quad (94) \]

depends on \( N - 1 \) variables \( \{q_j\} = \{q_1, \ldots, q_{N-1}\} \). In Eq. (93) the amplitude \( A_{N,B} \), whose computation is left in Appendix C, is given by

\[ A_{N,B} = \frac{2^{2N-N/2} N!}{\pi^{N/2+1}} \prod_{j=1}^N. \quad (95) \]

Moreover, the function \( \Theta_N \) defined by Eq. (94) can be written as a determinant: \( \Theta_N \) is a determinant, and the integration over \( q_N \) can be absorbed in its last line. The elements of this last line are indexed by the column index \( j \),

\[ \int_0^\infty dq_N q_N^j \cos \left(q_N - j\frac{\pi}{2}\right) e^{-\frac{q_N^2}{2M}} = \sqrt{\pi} M^{j+1} U_j(M, t), \quad (96) \]

with the help of the family of functions

\[ U_j(t) \equiv U_j(M, t) = i^{-\frac{j+1}{2}} H_j \left( \frac{M}{\sqrt{2t}} \right) e^{-\frac{t}{2}}, \quad (97) \]

where \( H_j \) is the Hermite polynomial of degree \( j \) [see formula (E2)]. Then \( \Theta_N(|q_j| |M, t) \) is the determinant of a matrix, whose elements indexed by \( (a, b) \) are given by

\[ q_a^{j+1} \cos \left(q_a - b\frac{\pi}{2}\right) \]

if \( 1 \leq a \leq N - 1 \) and \( 1 \leq b \leq N \),

\[ \sqrt{\pi} M^{j+1} U_a(M, t) \]

if \( a = N \) and \( 1 \leq b \leq N \).

The next step is to expand the determinant \( \Theta_N \) in minors with respect to its last line:

\[ \Theta_N(q_1, \ldots, q_{N-1} | M, t) = \sum_{j=1}^N (-1)^{N+j} \sqrt{\pi} M^{j+1} U_j(M, t) \det [M_{N,j} (\Theta_N)], \quad (98) \]

where \( \det [M_{N,j} (\Theta_N)] \) denotes the determinant of the minor \( (N, j) \) of the matrix \( \Theta_N \), obtained by removing its \( N \)th line and its \( j \)th column. These minors only involve the \( q_1, \ldots, q_{N-1} \) and \( M \), not \( \tau_M \). Expanding the product of the two functions \( \Theta_N \) entering into the expression (93) yields

\[ \Theta_N(|q_j| |M, \tau_M) \Theta_N(|q_j| |M, 1 - \tau_M) = \sum_{i,j=1}^N (-1)^{i+j} M^{i+j+2} U_i(\tau_M) U_j(1 - \tau_M) \times \det [M_{N,i} (\Theta_N)] \det [M_{N,j} (\Theta_N)], \quad (100) \]

The dependence on the variables \( q_j \) appears only in the last line of that expression (100), and the Cauchy-Binet formula (B7) makes it possible to compute the integration with respect to \( q_1, \ldots, q_{N-1} \) of the product of determinants of minors:

\[ \int_0^\infty dq_1, \ldots, dq_{N-1} e^{-\frac{\sum_{i=1}^{N-1} q_i^2}{2M}} \times \det [M_{N,i} (\Theta_N)] \det [M_{N,j} (\Theta_N)] = (N - 1)! \det \left[ M_{a,b} \left( M^{a+b-1} \sqrt{\frac{\pi}{2a+b+1}} (D_{N,B})_{a,b} \right) \right], \quad (101) \]

where \( a, b \) are, respectively, the line and column indices of the matrix of which we take the determinant, and \( D_{N,B} \) is the \( N \times N \) matrix with elements

\[ (D_{N,B})_{a,b} = (-1)^{a+b} H_{a+b-2}(0) - e^{-2M^2} H_{a+b-2}(\sqrt{2M}). \quad (102) \]

This matrix enters into the expression of the cumulative distribution of the maximal height of \( N \) watermelons [32]:

\[ \text{Prob} \left[ \max_{0 \leq \tau \leq 1} x_N(\tau) \leq M \right] = \frac{1}{\prod_{j=1}^{N-1} (j!^2)} \det D_{N,B}. \quad (103) \]

Putting together (100) and (101) in formula (93) and dividing by \( K_{N,B} \) given in Eq. (92), one finds the joint PDF of \( M \) and \( \tau_M \).

\[ P_{N,B}(M, \tau_M) = B_{N,B} \sum_{i,j=1}^N (-1)^{i+j} U_i(\tau_M) U_j(1 - \tau_M) \times \det [M_{i,j} (D_{N,B})] \quad (104a) \]

or, equivalently,

\[ P_{N,B}(M, \tau_M) = -B_{N,B} \det \left[ D_{N,B} \right] U(1 - \tau_M) D_{N,B}^{-1} U(\tau_M) \quad (104b) \]

with the normalization constant \( B_{N,B} = \sqrt{\sum_{j=1}^{N-1} j!^2} \).

It is also interesting to characterize the small \( \tau_M \) behavior of the distribution of \( P_{N,B}(\tau_M) = \int_0^\infty P_{N,B}(M, \tau_M) dM \). To study it, it is useful to start from Eq. (93), which, after integration over \( M \), yields
DISTRIBUTION OF THE TIME AT WHICH $N$ VICIOUS...
where \( \mathbf{n} \equiv (n_1, n_2, \ldots, n_N) \). One can then use the expansion to lowest order in \( \epsilon \)

\[
\det \psi_n(\epsilon) = (-1)^{N^{(N-1)}} \left( \prod_{j=1}^{N} \frac{j^{2j-1}}{(2j-1)!} \right) 2^N \pi^{N^2} \left( \frac{1}{M} \right)^{N(N+1)} 
\times \prod_{i=1}^{N} n_i \Delta_N(n_i^2, \ldots, n_N^2)e^{N^2} + o(\epsilon^{N^2}),
\]

(116)

where \( \Delta_N(\lambda_1, \ldots, \lambda_N) \) is the \( N \times N \) Vandermonde determinant. In this formula the first line contains only numerical constants, irrelevant for the computation (they will be absorbed in the normalization constant), so only the second line is relevant, containing the dependence in \( M \) and in the indices of the sums \( n_i \) and the lowest order in \( \epsilon^{N^2} \). Hence, one has to lowest order

\[
W_{N,E}(M - \eta, \tau_M, \epsilon) = \eta^2 \epsilon^{2N^2} W_{N,E}(M, \tau_M) + O(\eta^3 \epsilon^{2N^2}, \eta^2 \epsilon^{2N^2+1}),
\]

(117)

as well as, for the normalization,

\[
Z_{N,E}(\eta, \epsilon) = \eta^2 \epsilon^{2N^2} K_{N,E} + o(\eta^2 \epsilon^{2N^2}),
\]

(118)

with \( K_{N,E} \) a number. Taking the double limit \( \epsilon, \eta \to 0 \) one obtains the joint probability

\[
P_{N,E}(M, \tau_M) = \frac{W_{N,E}(M, \tau_M)}{K_{N,E}} = \frac{A_{N,E}}{M^{N(N+1)+3}}
\times \sum_{n_n} \left\{ (-1)^{n^2 + n_n^2} n_n^2 n_n^2 \prod_{i=1}^{N-1} n_i^2 \Delta_N(n_i^2, \ldots, n_{N-1}^2) \right\}
\times \Delta_N(n_1^2, \ldots, n_{N-1}^2, n_N^2) e^{-\frac{\pi^2}{3M} \sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{3M} \sum_{i=1}^{N-1} \omega_i(t) n_i^2},
\]

(119)

where the numerical constant is given by (see Appendix C 2)

\[
A_{N,E} = \frac{N \pi^{2N^2+N+3}}{2^{2N^2-N/2} \prod_{j=1}^{N-1} \Gamma(2j+1) \Gamma\left( \frac{3}{2} + j \right)}.
\]

(120)

Following the notations introduced in the \( N = 2 \) case, we define [writing for compactness the \( N - 1 \)-plet \( \{n_i\} = (n_1, \ldots, n_{N-1}) \)]

\[
\Omega_p(\{n_i\}, \tau_M) = \sum_{k=1}^{\infty} (-1)^k k^2 \Delta_N(n_1^k, k^2) e^{-\frac{\pi^2}{3M} n_1^2 k^2} \epsilon^{(N-1)k^2}.
\]

(121)

where we used the property of the Vandermonde determinant to factorize the sum with respect to \( k \) in the last line of the determinant, which plays the role of \( n_N \) associated with \( t = \tau_M \) or the role of \( n_N' \) with \( t = 1 - \tau_M \), resulting in elements such as \( \omega_i(t) \equiv \omega_i(M, t) \), for \( 1 \leq i \leq N \),

\[
\omega_i(t) = \sum_{k=1}^{\infty} (-1)^k k^2 e^{-\frac{\pi^2}{3M} n_1^2 k^2}.
\]

(122)

With this notation, we obtain the compact formula

\[
P_{N,E}(M, \tau_M) = \frac{A_{N,E}}{M^{N(N+1)+3}} \prod_{i=1}^{N} \left\{ \sum_{n_i} n_i^2 e^{-\frac{\pi^2}{3M} n_i^2} \right\}
\times \Omega_N(\{n_i\}|M, \tau_M) \times \Omega_N(\{n_i\}|M, 1 - \tau_M),
\]

(123)

which is the “discrete” analog of formula (93) obtained in the case of \( N \) bridges.

The ultimate step is to expand each determinant \( \Omega_N \) with respect to their last line. Then, using the discrete Cauchy-Binet identity with respect to the sums over \( n_i \), one has

\[
P_{N,E}(M, \tau_M) = B_{N,E} \frac{1}{M^2} \sum_{i,j} \left( \frac{2 \tau^2}{M^2} \right)^{i+j} \omega_i(\tau_M) \omega_j(1 - \tau_M) det[M_{i,j}(D_{N,E})],
\]

(124)

with the normalization

\[
B_{N,E} = \frac{(-1)^{N^2 + 3/2} \sqrt{2} \pi^{2N^2}}{\prod_{j=1}^{N-1} \Gamma\left( \frac{3}{2} + j \right)}.
\]

(125)

and where appear the minors of the matrix \( D_{N,E} \equiv D_{N,E}(M) \) whose elements are, for \( 1 \leq i, j \leq N \),

\[
D_{N,E,i,j} = \sum_{n=-\infty}^{\infty} \frac{H_{2(i+j-1)}(\sqrt{2}M)n e^{-2M^2n^2}}{2^i 2^j \Gamma(2i+1) \Gamma(2j+1)}.
\]

(126)

This matrix enters into the expression of the cumulative probability of the maximum [29]:

\[
\text{Prob} \left[ \max_{0 \leq \tau \leq 1} S_N(\tau) \leq M \right] = (-1)^N \frac{\det D_{N,E}}{2^{N(N+1)} \prod_{j=1}^{N} (2j-1)}.
\]

(127)

Defining the vector with elements, for \( 1 \leq i \leq N \),

\[
U_{E,i}(t) \equiv U_{E,i}(M,t) = \frac{1}{M} \left( \frac{2 \tau^2}{M^2} \right)^i \omega_i(t),
\]

(128)

one is able to express the result as the determinant

\[
P_{N,E}(M, \tau_M) = -B_{N,E} \det \left( \begin{array}{cc} D_{N,E} & U_{E}(\tau_M) \\ \tau E(1 - \tau_M) & 0 \end{array} \right),
\]

(129a)

or as the matrix product

\[
P_{N,E}(M, \tau_M) \equiv B_{N,E} \det[D_{N,E}] U_{E}(1 - \tau_M) D_{N,E}^{-1} U_{E}(\tau_M).
\]

(129b)

The behavior of the marginal distribution \( P_{N,E}(\tau_M) \) when \( \tau_M \to 0 \) can be obtained as explained in Sec. II G for \( N = 2 \)
vicious excursions. The leading behavior, when $\tau_M \to 0$, is given by

$$
P_{N,E}(\tau_M) \sim e^{-\frac{\tau_M}{D}} \sqrt{\frac{N! \pi (2N-1)}{16}}. 
$$

However, the algebraic correction to this leading behavior is hard to evaluate.

IV. NUMERICAL SIMULATIONS

In this section we present the results of our numerical simulations of the distribution $P_{N,E}(\tau_M)$ both for the cases of bridges and excursions. To generate numerically such watermelon configurations we exploit the connection between these vicious walkers problems and Dyson’s BM, which we first recall.

A. Relationship with Dyson’s Brownian motion

To make the connection between the vicious walkers problem and Dyson’s BM, we consider the propagator of the $N$ vicious walkers,

$$
P_N(v,t_2|u,t_1) \equiv P_N(v_1,\ldots,v_N,t_2|u_1,\ldots,u_N,t_1),
$$

(131)

which is the probability that the process reaches the configuration $x_1(t_2) = v_1, \ldots, x_N(t_2) = v_N$ at time $t_2$ given that $x_1(t_1) = u_1, \ldots, x_N(t_1) = u_N$ at time $t_1$: This is thus a conditional probability. This propagator satisfies the Fokker-Planck equation,

$$
\frac{\partial}{\partial t_2} P_N(v,t_2|u,t_1) = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial v_i^2} P_N(v,t_2|u,t_1),
$$

(132)

together with the initial condition,

$$
P_N(v,t_2|u,t_1) = \delta^{(N)}(v - u),
$$

(133)

and the noncrossing condition, which is specific to this problem,

$$
P_N(v,t_2|u,t_1) = 0, \text{ if } u_i = v_j, \forall (t_1,t_2).
$$

(134)

On the other hand, let us focus on Dyson’s BM. For this purpose we consider random matrices whose elements are time dependent and are themselves BMs. For instance, for random matrices from GUE, that is, with $\beta = 2$, we consider random Hermitian matrices $H \equiv H(t)$, of size $N \times N$, which elements $H_{mn}(t)$ are given by

$$
H_{mn}(t) = \begin{cases} 
\frac{1}{2\sqrt{2}}(b_{mn}(t) + i \tilde{b}_{mn}(t)), & m < n, \\
\bar{b}_{mn}(t), & m = n, \\
\frac{1}{2\sqrt{2}}(b_{mn}(t) - i \tilde{b}_{mn}(t)), & m > n,
\end{cases}
$$

(135)

where $b_{mn}(t)$ and $\tilde{b}_{mn}(t)$ are independent BMs (with a diffusion constant $D = 1/2$). We denote $\lambda_1(t) < \lambda_2(t) < \cdots < \lambda_N(t)$ the $N$ eigenvalues of $H(t)$ (135) [or more generally of a matrix belonging to a $\beta$ ensemble with $\beta = 1,2,4$ which is constructed as in Eq. (135) with the appropriate symmetry]. One can then show that the $\lambda_i$’s obey the following equations of motion (which define Dyson’s BM) [41,42]:

$$
\frac{d\lambda_i(t)}{dt} = \frac{\beta}{2} \sum_{1 \leq j \neq i \leq N} \frac{1}{\lambda_i(t) - \lambda_j(t)} + \eta_i(t),
$$

(136)

where $\eta_i$’s are independent Gaussian white noises, $\langle \eta_i(t)\eta_i(t') \rangle = \delta(t-t')$. Let $P_{\text{Dyson}}(\lambda,t|\mu,0) \equiv P_{\text{Dyson}}(\lambda_1,\ldots,\lambda_N,t|\mu_1,\ldots,\mu_N,t = 0)$ be the propagator of this Dyson’s BM (136). It satisfies the Fokker-Planck equation

$$
\frac{\partial}{\partial t} P_{\text{Dyson}} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} P_{\text{Dyson}} - \frac{\beta}{2} \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \left[ \sum_{1 \leq j \neq i \leq N} \frac{1}{\lambda_i - \lambda_j} P_{\text{Dyson}} \right].
$$

(137)

Of course, one has also here $P_{\text{Dyson}}(\lambda,t|\mu,0) = 0$ if $\lambda_i = \lambda_j$, for any time $t$ and from Eq. (137) one actually obtains

$$
P_{\text{Dyson}}(\lambda,t|\mu,0) \sim (\lambda_j - \lambda_i)^{\beta}, \lambda_i \to \lambda_j.
$$

(138)

This Fokker-Planck equation can be transformed into a Schrödinger equation by applying the standard transformation [75]:

$$
P_{\text{Dyson}}(\lambda,t|\mu,0) = \frac{\exp \left[ \frac{\beta}{2} \sum_{1 \leq j < \lambda_N < N} \log (\lambda - \lambda_j) \right]}{\exp \left[ \frac{\beta}{2} \sum_{1 \leq j < \mu_N < N} \log (\mu - \mu_j) \right]} \times P_{\text{Dyson}}(\lambda,t|\mu,0),
$$

(139)

where $P_{\text{Dyson}}(\lambda,t|\mu,0)$ is such that

$$
P_{\text{Dyson}}(\lambda,t = 0|\mu,0) = \delta^{(N)}(\lambda - \mu).
$$

(140)

On the other hand, given Eqs. (138) and (139), one has

$$
P_{\text{Dyson}}(\lambda,t|\mu,0) \sim (\lambda_j - \lambda_i)^{\beta/2}, \lambda_i \to \lambda_j.
$$

(141)

From Eqs. (137) and (139), one obtains that $P_{\text{Dyson}}$ satisfies the following Schrödinger equation:

$$
\frac{\partial}{\partial t} W_{\text{Dyson}} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} W_{\text{Dyson}} - \frac{\beta}{8} (\beta - 2) \sum_{i=1}^{N} \sum_{1 \leq j \neq i \leq N} \frac{1}{(\lambda_j - \lambda_i)^2} W_{\text{Dyson}}.
$$

(142)

Hence, for a generic value of $\beta$, (142) is the Schrödinger equation associated with a Calogero-Sutherland Hamiltonian (on an infinite line) [76,77]. However, for the special value $\beta = 2$, the strength of the interaction vanishes exactly and in that case the equation satisfied by $W_{\text{Dyson}}$ [Eqs. (138), (140), and (142)] is identical to the one satisfied by the propagator in the vicious walkers problem [Eqs. (132)–(134)]: In that case, Eq. (142) corresponds to the Schrödinger equation for free fermions. In other words, for the special value $\beta = 2$ there exists a simple relation between vicious walkers and Dyson BM which can be simply written as

$$
P_{\text{Dyson}}(\lambda,t|\mu,0) = \frac{\prod_{j=1}^{N} (\lambda_j - \lambda_i)}{\prod_{j=1}^{N} (\mu_j - \mu_i)} P_N(\lambda,t|\mu,0).
$$

(143)

In particular, for watermelons without wall on the unit time interval, for which the initial and final positions coincide, this relation (143) tells us that this process is identical to
Dyson’s BM where the elements of the matrices are themselves Brownian bridges, that is, \(b_{ij} \to B_{ij}, \tilde{b}_{ij} \to \tilde{B}_{ij}\) in Eq. (135) such that \(B_{ij}(0) = B_{ij}(1) = 0\) and \(\tilde{B}_{ij}(0) = \tilde{B}_{ij}(1) = 0\). This then makes it possible to generate easily watermelons because one can generate a Brownian bridge \(B_{ij}(t)\) on the interval \([0,1]\) from a standard BM \(b_{ij}(t)\) via the relation \(B_{ij}(t) = b_{ij}(t) - t b_{ij}(1)\).

For nonintersecting excursions, one can show that the dynamics corresponds to Dyson’s BM associated with random symplectic and Hermitian matrices, as explained in Ref. [29].

**B. Numerical results**

To sample nonintersecting Brownian bridges we use the aforementioned equivalence to Dyson’s BM. We consider the discrete-time version of the matrix in Eq. (135) at step \(0 \leq k \leq T\):

\[
H_{i,j}(k) = \begin{cases} 
\frac{1}{\sqrt{2}}[B_{ij}(k) + i B_{ij}(k)] & 1 \leq i < j \leq N, \\
B_{ii}(k) & i = j, 1 \leq i \leq N, \\
\frac{1}{\sqrt{2}}[B_{ij}(k) - i B_{ij}(k)] & 1 \leq i < j \leq N,
\end{cases}
\]

(144)

where \(B_{ij}(k)\) are independent discrete-time Brownian bridges. They are constructed from ordinary discrete-time and continuous space random walks \(b_{ij}(k)\) where each jump is drawn from a Gaussian distribution. The aforementioned relation, to construct Brownian bridges, reads in discrete time \(B_{ij}(k) = b_{ij}(k) - (k/T) b_{ij}(T)\).

Diagonolizing the matrix \(H_{i,j}(k)\) at each step \(k\) gives the ordered ensemble of \(N\) distinct eigenvalues \(\lambda_1(k) < \lambda_2(k) < \cdots < \lambda_N(k)\), which is equivalent to a sample of nonintersecting Brownian bridges [see Eq. (143)]. For each sample, we keep in memory the maximal value of \(\lambda_N(k)\), and the step time \(k_M\) at which \(\lambda_N(k_M)\), which is \(\max_{0 \leq k \leq T} \lambda_N(k)\). The plots shown here are obtained by making the histogram of the rescaled time \(k_M/T\) (with a bin of size 8 to smooth the curve), with \(T = 256\) steps averaged over \(10^6\) samples, for \(N = 2,3,\ldots,16\). For

![Fig. 10.](image)

**FIG. 10.** (Color online) Plots of the distribution \(P_{N,B}(\tau_M)\) of the time \(\tau_M\) to reach the maximum for \(N = 2,3,4,5,10\) nonintersecting Brownian bridges. The dots correspond to our numerical data while the solid lines correspond to our analytical predictions. There is no fitting parameter.

Brownian bridges, that is, \(b_{ij} \to B_{ij}, \tilde{b}_{ij} \to \tilde{B}_{ij}\) in Eq. (135) such that \(B_{ij}(0) = B_{ij}(1) = 0\) and \(\tilde{B}_{ij}(0) = \tilde{B}_{ij}(1) = 0\). This then makes it possible to generate easily watermelons because one can generate a Brownian bridge \(B_{ij}(t)\) on the interval \([0,1]\) from a standard BM \(b_{ij}(t)\) via the relation \(B_{ij}(t) = b_{ij}(t) - t b_{ij}(1)\).

For nonintersecting excursions, one can show that the dynamics corresponds to Dyson’s BM associated with random symplectic and Hermitian matrices, as explained in Ref. [29].

First we present the comparison between our exact result for Brownian bridges and numerical simulations for \(N = 2,3,4,5,10\). The agreement between our numerical results and our exact analytical formula is perfect (see Fig. 10). Then, anticipating the application of our results to growing interfaces, we show a plot of the rescaled distribution as a function of the rescaled time \(\tau_M = (\tau_M - 1/2)/N^{1/3}\). Despite the fact that this scaling requires in principle that \(N \gg 1\), one sees that the rescaled distributions for moderate values of \(N = 10,12,14,16\) fall on a single master curve (see Fig. 11). However, given the relatively narrow range of \(N\) explored in these simulations, one cannot be sure that the asymptotic, large \(N\), behavior is reached.

To compare our analytical results for \(N\) watermelons to growing interfaces models and the DPRM, we have simulated both the PNG model and a discrete model of DPRM on a lattice. To sample the PNG model, we have applied the rules that were discussed before (see Fig. 1) where the time interval is discretized. In particular, we use a discretization of the nucleation process: At each time step, a nucleation occurs at site \(x\) such that \(|x| < t\) with a small probability \(p\), whose value is chosen such that one recovers a uniform density of nucleation events \(\rho = 2\) in the continuum limit. The interface evolves during a time \(T\). We compute the distribution of the position \(X_M\) of the maximal height of the droplet [see Fig. 2(a)], averaged over \(10^5\) samples. In Fig. 12 we have plotted the distribution of the rescaled position \(u_M = X_M / T^{3/2}\) for \(T = 64\) and \(T = 90\) (both of them being statistically independent): The collapse of these data on a single master curve is consistent with the expected KPZ scaling in \(T^{3/2}\).

On the other hand, we have simulated a directed polymer model in 1 + 1 dimensions as depicted in Fig. 3(a), where on each site, there is a random energy variable, with Gaussian fluctuations. We find the optimal path of length \(L\), which is the polymer with minimal energy, with one free end point. We compute the distribution of the position \(X_M\) of this end point, averaged over \(10^5\) samples. In Fig. 12 we
Brownian motion, which uses the deep connection between vicious walkers and RMT, which we have recalled in detail.

Thanks to the connection between vicious walkers and the Airy process (7), our results yield, in the large \(N\) limit, results for the distribution of \(X_M\), the position of the maximal height of a curved growing interface in the KPZ universality class [Fig. 2(a)] or the transverse coordinate of the end point of the optimal directed polymer [Fig. 2(b)]. We have shown that our analytical results for finite \(N \sim 15\), correctly rescaled, are in good agreement with our numerical data for the PNG model and DPRM on the square lattice. Performing the large \(N\) asymptotic analysis of our formula [Eqs. (93) and (123)] remains a formidable challenge, which will hopefully motivate further study of this problem.

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APPENDIX A: INTEGRATION OVER ORDERED VARIABLES

In the main text, we have to sum over realizations with the constraint \(x_1 < x_2 < \cdots < x_{N-1} < x_N\). Here we study how integrals may be simplified by the symmetry of the integrand.

1. \(N = 2\), two variables

Consider a box \([a, b]\) in one dimension, and a function of two variables \(f(x_1, x_2)\). In the main text, we want to compute ordered integrals such as

\[
\int_a^b \int_a^{x_2} dx_1 f(x_1, x_2).
\]

(A1)

It is usually easier to compute the integral over the whole domain; that is to say

\[
\int_a^b dx_2 \int_a^b dx_1 f(x_1, x_2).
\]

(A2)

Here we make a link between those two quantities, when \(f\) is symmetric in the exchange of \(x_1\) and \(x_2\): \(f(x_2, x_1) = f(x_1, x_2)\). Let us start from Eq. (A2) with a symmetric function \(f\)

\[
\int_a^b dx_2 \int_a^b dx_1 f(x_1, x_2)
= \int_a^b dx_2 \int_a^{x_2} dx_1 f(x_1, x_2) + \int_a^b dx_2 \int_{x_2}^b dx_1 f(x_1, x_2)
= \int_a^b dx_2 \int_a^{x_2} dx_1 f(x_1, x_2) + \int_a^{x_1} dx_1 \int_a^b dx_2 f(x_1, x_2)
= \int_a^b dx_2 \int_a^{x_2} dx_1 f(x_1, x_2) + \int_a^{x_2} dx_2 f(x_2, x_1)
= 2 \int_a^b dx_2 \int_a^{x_2} dx_1 f(x_1, x_2).
\]

(A3)

From the second to the third line we use triangular integration

\[
\{a < x_2 < b, x_2 < x_1 < b\} = \{a < x_1 < b, a < x_2 < x_1\}
\]

in

\[
\{a < x_2 < b, x_2 < x_1 < b\} = \{a < x_1 < b, a < x_2 < x_1\}
\]

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the second term. From the third line to the forth, we relabeled the variables in the second term. In this operation if $\epsilon$ was antisymmetric, we would have a minus sign, and hence a cancellation of the two terms. The conclusion for two variables is that if $\epsilon$ is antisymmetric, the integration over the whole domain is zero, but if $\epsilon$ is symmetric in the exchange of the two variables, we have the useful formula

$$\int_a^b dx_2 \int_a^{x_2} dx_1 f(x_1, x_2) = \frac{1}{2} \int_a^b dx_2 \int_a^{x_2} dx_1 f(x_1, x_2).$$

(A4)

$N$ variables

The previous result can be extended for $N$ variables, and the result is, for $f(x_1, \ldots, x_N)$, symmetric in exchange of any two of its arguments,

$$\int_{\text{ord}} dx \ f(x) = 2^{-\frac{N(N-1)}{2}} \int dx \ f(x),$$

(A5)

with $x = (x_1, \ldots, x_N)$ and an integration on the domain $a < x_1 < x_2 < \ldots < x_N < b$ in the left-hand side, and an integration over the whole domain $a < x_i < b$ for $1 \leq i \leq N$ in the right-hand side.

The proof is done by applying iteratively the formula (A4) first to the $N-1$ couples $(x_N, x_{N-1})$, $(x_N, x_{N-2}), \ldots, (x_N, x_1)$ changing the upper limit of integrations from $b$ to $x_N$. This operation gives a factor of 2 for each couple so that

$$\int_a^b dx_N \int_a^{x_N} dx_{N-1} \cdots \int_a^{x_N} dx_2 \int_a^{x_N} dx_1 f(x)$$

$$= 2^{N-1} \int_a^b dx_N \int_a^{x_N} dx_{N-1} \cdots \int_a^{x_N} dx_2 \int_a^{x_N} dx_1 f(x).$$

(A6)

Then one repeats this procedure for the $N-2$ couples $(x_{N-1}, x_{N-2})$ to $(x_{N-1}, x_1)$, which gives a factor $2^{N-2}$, and continues until step $k$ with the $N-k$ couples $(x_{N-k+1}, x_{N-k})$ to $(x_{N-k+1}, x_1)$ to have

$$\int_a^b dx_N \int_a^{x_N} dx_{N-1} \cdots \int_a^{x_N} dx_2 \int_a^{x_N} dx_1 f(x)$$

$$= 2^{N-1} 2^{N-2} \cdots 2^{N-k}$$

$$\times \int_a^b dx_N \int_a^{x_N} dx_{N-1} \cdots \int_a^{x_N} dx_{N-k} \int_a^{x_N} dx_1 f(x)$$

(A7)

Doing so until $k = N - 1$ one obtains the result (A5).

**APPENDIX B: CAUCHY-BINET IDENTITY**

1. Continuous version

In the main text, we are confronted with integrals of two determinants that take the form

$$I^{(N)} = \int dx \ \det_{1 \leq i, j \leq N} [f_i(x_j)] \ \det_{1 \leq k, l \leq N} [g_{kl}(x_i)].$$

(B1)

where $dx = \prod_{i=1}^N dx_i$, $f_i, g_{kl}$, $1 \leq i \leq N$ are functions of one variable, and each variable $x_i$ has the same domain of integration (in the main text we usually integrate over the momenta $k_i$ from 0 to $+\infty$). We have

$$I^{(N)} = \int d\sigma \ \prod_{i=1}^N \int dx_i \ \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{i=1}^N f_i(x_{\sigma(i)})$$

$$\times \left[ \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{i=1}^N g_{\sigma^{-1}(i)}(x_{\sigma(i)}) \right].$$

(B2)

$S_N$ is the symmetric group, $\epsilon(\sigma)$ is the signature of the permutation $\sigma \in S_N$. Now we make the change of variables $x'_i = x_{\sigma(i)}$. That is, $x_j = x'_{\sigma^{-1}(j)}$ and the Jacobian is unity. So we have

$$I^{(N)} = \sum_{\sigma \in S_N} \epsilon(\sigma) \int \prod_{i=1}^N dx'_i \ \prod_{i=1}^N f_i(x'_i)$$

$$\times \left[ \sum_{\sigma \in S_N} \epsilon(\sigma') \prod_{i=1}^N g_{\sigma^{-1}(i)}(x'_{\sigma^{-1}(i)}) \right].$$

(B3)

$S_N$ is a group, so it exists $\sigma'' \in S_N$ so that $\sigma'' = \sigma^{-1} \circ \sigma'$. and $\epsilon(\sigma') = \epsilon(\sigma^{-1}) \epsilon(\sigma) = \epsilon(\sigma) \epsilon(\sigma')$. Moreover, $\sum_{\sigma \in S_N} = \sum_{\sigma^{-1} \in \sigma \in S_N} = \sum_{\sigma'' \in S_N}$. Getting back to the $x_i$ variables we have

$$I^{(N)} = \sum_{\sigma \in S_N} \epsilon(\sigma) \int \prod_{i=1}^N dx_i \ \prod_{i=1}^N f_i(x_i)$$

$$\times \left[ \sum_{\sigma \in S_N} \epsilon(\sigma') \prod_{i=1}^N g_{\sigma^{-1}(i)}(x_{\sigma^{-1}(i)}) \right]$$

$$= \sum_{\sigma \in S_N} \int \prod_{i=1}^N dx_i \ \prod_{i=1}^N f_i(x_i)$$

$$\times \left[ \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{i=1}^N g_{\sigma^{-1}(i)}(x_{\sigma^{-1}(i)}) \right]$$

(B4)

$$= \sum_{\sigma \in S_N} \int \prod_{i=1}^N dx_i \ \prod_{i=1}^N f_i(x_i)$$

$$\times \left[ \sum_{\sigma \in S_N} \epsilon(\sigma') \prod_{i=1}^N g_{\sigma^{-1}(i)}(x'_{\sigma^{-1}(i)}) \right].$$

(B5)
In the last line, the term in brackets is simply the determinant \( \det_{1 \leq k, l \leq N} \{ g_k(x_l) \} \). The permutation \( \sigma \) does not appear anymore, so we have \( \sum_{\sigma \in S_N} 1 = \text{Card}(S_N) = N! \). Hence,

\[
I^{(N)} = N! \int \prod_{i=1}^{N} dx_i \prod_{k=1}^{N} f_k(x_k) \det_{1 \leq x_i \leq x_k \leq N} \{ g(x_i) \}.
\]

At this stage, the basic result is that we have expressed the first determinant only as a product of the diagonal terms. This is used in the main text.

To go further we use the fact that the determinant of one matrix is the same as the determinant of its transpose:

\[
I^{(N)} = N! \int \prod_{i=1}^{N} dx_i \prod_{k=1}^{N} f_k(x_k) \times \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{k=1}^{N} g_{\sigma(k)}(x_k)
\]

\[
= N! \sum_{\sigma \in S_N} \epsilon(\sigma) \int \prod_{i=1}^{N} dx_i \prod_{k=1}^{N} f_k(x_k) g_{\sigma(k)}(x_k)
\]

\[
= N! \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{k=1}^{N} \left\{ \int dx f_k(x) g_{\sigma(k)}(x) \right\}.
\]

(B6)

From the second to the third line, we factorized the integrals, and we wrote \( x = x_k \) for each \( k \). We recognize the determinant of the matrix whose elements indexed by \((i,j)\) are given by an integral of the product \( f_i \times g_j \), yielding the so-called Cauchy-Binet identity:

\[
\int dx \det_{1 \leq i, j \leq N} \{ f_i(x_j) \} \det_{1 \leq k, l \leq N} \{ g_k(x_l) \} = N! \det_{1 \leq i, j \leq N} \left\{ \int dx f_i(x) g_j(x) \right\}.
\]

(B7)

Moreover, one can easily deduce from Eq. (B7) the following relation, valid for any integrable function \( h(x) \):

\[
\int dx \prod_{i=1}^{N} h(x_i) \det_{1 \leq i, j \leq p} \{ f_i(x_j) \} \det_{1 \leq i, j \leq p} \{ g_i(x_j) \} = N! \det_{1 \leq i, j \leq N} \left\{ \int dx h(x) f_i(x) g_j(x) \right\}.
\]

(B8)

This relation is used in the main text.

2. Discrete version

Here we consider the following multiple sum:

\[
I_N = \sum_{n_1} \sum_{n_2} \ldots \sum_{n_N} \det_{1 \leq i, j \leq N} \{ f_i(n_j) \} \det_{1 \leq i, j \leq N} \{ g_i(n_j) \}
\]

\[
= \sum_{\sigma \in S_N} \epsilon(\sigma) \sum_{n_1} \sum_{n_2} \ldots \sum_{n_N} \prod_{i=1}^{N} f_i(n_{\sigma(i)})
\]

\[
\times \det_{1 \leq i, j \leq N} \{ g_i(n_j) \}.
\]

(B9)

Changing the indices of the sums \( m_i = n_{\sigma(i)} \) or \( n_k = m_{\sigma^{-1}(k)} \), one gets

\[
I_N = \sum_{\sigma \in S_N} \epsilon(\sigma) \sum_{m_1} \sum_{m_2} \ldots \sum_{m_N} \prod_{i=1}^{N} f_i(m_i)
\]

\[
\times \det_{1 \leq i, j \leq N} \{ g_i(m_{\sigma^{-1}(j)}) \}
\]

\[
= \sum_{\sigma \in S_N} \epsilon(\sigma) \sum_{m_1} \sum_{m_2} \ldots \sum_{m_N} \prod_{i=1}^{N} f_i(m_i)
\]

\[
\times \epsilon(\sigma) \det_{1 \leq i, j \leq N} \{ g_i(m_j) \},
\]

(B10)

where we rearranged the columns in the remaining determinant in the “natural” order, thus factorizing the signature of the permutation \( \sigma \). Hence, there is a sum over the permutations \( \sigma \in S_N \) of \( \epsilon^2(\sigma) \), giving the cardinal \( N! \). We obtain as in the continuous case that the first determinant is written as the product of its diagonal terms with a factor \( N! \):

\[
I_N = N! \sum_{m_1} \sum_{m_2} \ldots \sum_{m_N} \prod_{i=1}^{N} f_i(m_i)
\]

\[
\times \det_{1 \leq i, j \leq N} \{ g_i(m_j) \}
\]

\[
= N! \sum_{m_1} \sum_{m_2} \ldots \sum_{m_N} \prod_{i=1}^{N} f_i(m_i)
\]

\[
\times \epsilon(\sigma) \det_{1 \leq i, j \leq N} \{ g_i(m_j) \},
\]

(B11)

Factorizing the sums, one gets

\[
I_N = N! \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{i=1}^{N} \{ f_i(m) g_{\sigma(i)}(m) \}
\]

\[
= N! \det_{1 \leq i, j \leq N} \left( \sum_{m} f_i(m) g_j(m) \right).
\]

(B12)

In the same way as in the continuous case, if one considers

\[
J_N = \sum_{n_1} \sum_{n_2} \ldots \sum_{n_N} \prod_{i=1}^{N} h(n_i)
\]

\[
\times \det_{1 \leq i, j \leq N} \{ f_i(n_j) \} \det_{1 \leq i, j \leq N} \{ g_i(n_j) \},
\]

one finally obtains

\[
J_N = N! \det_{1 \leq i, j \leq N} \left( \sum_{m} h(m) f_i(m) g_j(m) \right).
\]

(B14)

APPENDIX C: NORMALIZATION CONSTANT

In this section, we compute the normalization constant of the joint distributions \( P_{N,B}(M,\tau_M) \) and \( P_{N,E}(M,\tau_M) \) for nonintersecting bridges and excursions, respectively.

1. Nonintersecting Brownian bridges

To compute the amplitude \( A_{N,B} \) in Eq. (95) we use the fact that the integration of \( P_{N,B}(M,\tau_M) \) in Eq. (93), that is,
that given by
\[ F_{N,B}(M) = \text{Prob}\left[ \max_{0 \leq \tau \leq 1} x_N(\tau) \leq M \right] \tag{C1} \]
and
\[ a_{N,B} = \frac{2^{2N}}{(2\pi)^{N/2}} \prod_{j=1}^{N} j!, \tag{C3} \]
where the amplitude \( a_{N,B} \) can be computed using a Selberg integral \cite{10.2307/75799}

The PDF of the maximum, \( F'_{N,B}(M) = \partial_M F_{N,B}(M) \) is thus given by
\[ F'_{N,B}(M) = -N^2 \frac{a_{N,B}}{M^{N+1}} \int_0^\infty dq \ e^{-\frac{q^2}{M^2}} \Theta_N(q)^2 \tag{C4} \]
and
\[ + \frac{N a_{N,B}}{M^{N+1}} \int_0^\infty dq \ e^{-\frac{q^2}{M^2}} \sum_{j,l=1}^{N} (-1)^{j+l} q_N^{j+l} \cos \left( q_N - j\frac{\pi}{2} \right) \]

one obtains finally
\[ F'_{N,B}(M) = -N a_{N,B} \frac{a_{N,B}}{M^{N+1}} \int_0^\infty dq \ e^{-\frac{q^2}{M^2}} \sum_{j,l=1}^{N} (-1)^{j+l} q_N^{j+l-1} \]
\[ \times \det[M_{j,N}(\Theta_N)] \det[M_{l,N}(\Theta_N)] \]
\[ \times \sin \left( 2q_N - (l + j)\frac{\pi}{2} \right). \tag{C7} \]

On the other hand, from the definition of \( P_{N,B}(M,\tau_M) \) given in Eq. (93) one has
\[ \int_0^1 P_{N,B}(M,\tau_M)d\tau_M = F'_{N,B}(M), \tag{C8} \]
where, in the last two lines, we have expanded both determinants in minors with respect to the last column [we recall that \( M_{j,N}(\Theta_N) \) denotes the minor \((j,N)\) of the matrix \( \Theta_N \), obtained by removing its \( j \)th line and its \( N \)th column]. Note that these minors do not depend on \( q_N \). This suggests to consider, in the last two lines of Eq. (C4), only the integral over the variable \( q_N \), where we perform an integration by part. This yields
\[ \int_0^\infty dq_N \frac{q_N}{M^2} e^{-\frac{q_N^2}{M^2}} \sin \left( q_N - j\frac{\pi}{2} \right) \cos \left( q_N - l\frac{\pi}{2} \right) \]
\[ = \left( j + l - 1 \right) \int_0^\infty dq_N q_N^{j+l-2} \cos \left( q_N - j\frac{\pi}{2} \right) \]
\[ \times \cos \left( q_N - l\frac{\pi}{2} \right) e^{-\frac{q_N^2}{2M^2}} + \int_0^\infty dq_N q_N^{j+l-1} e^{-\frac{q_N^2}{2M^2}} \]
\[ \times \partial_{q_N} \left[ \cos \left( q_N - j\frac{\pi}{2} \right) \cos \left( q_N - l\frac{\pi}{2} \right) \right]. \tag{C6} \]

Inserting this result of the integration by part (C6) in Eq. (C4) and using the identity (which we have explicitly checked for \( N = 2,3,4,5 \) but can only conjecture for any integer \( N > 5 \))

\[
\begin{vmatrix}
\sin(q_1) & \cdots & \sin(q_N) \\
q_1 \cos(q_1) & \cdots & q_N \cos(q_N) \\
q_1^2 \sin(q_1) & \cdots & q_N^2 \sin(q_N) \\
\vdots & \ddots & \vdots \\
q_1^{N-1} \cos(q_1 - N\frac{\pi}{2}) & \cdots & q_N^{N-1} \cos(q_N - N\frac{\pi}{2})
\end{vmatrix}
\times
\begin{vmatrix}
\sin(q_1) & \cdots & \sin(q_N) \\
q_1 \cos(q_1) & \cdots & 2q_N \cos(q_N) \\
q_1^2 \sin(q_1) & \cdots & 3q_N^2 \cos(q_N) \\
\vdots & \ddots & \vdots \\
q_1^{N-1} \cos(q_1 - N\frac{\pi}{2}) & \cdots & Nq_N^{N-1} \cos(q_N - N\frac{\pi}{2})
\end{vmatrix}
\]

with
\[ P_{N,B}(M,\tau_M) = \frac{A_{N,B}}{M^{N+3}} \int_0^\infty dq_1, \ldots, dq_{N-1} e^{-\frac{q_{N-1}^2}{2M^2}} \]
\[ \times \gamma_N((q_1)M,\tau_M)\gamma_N((q_1)|M,1 - \tau_M). \tag{C9} \]

To compute this integral over \( \tau_M \) (C8) we compute instead
\[ P(M,T) = \frac{A_{N,B}}{M^{N+3}} \int_0^T d\tau_M \int_0^\infty dq_1, \ldots, dq_{N-1} e^{-\frac{q_{N-1}^2}{2M^2}} \]
\[ \times \gamma_N((q_1)M,\tau_M)\gamma_N((q_1)|M,T - \tau_M), \tag{C10} \]
and clearly
\[ \int_0^1 P_{N,B}(M,\tau_M)d\tau_M = P(M,T = 1). \tag{C11} \]
The structure of $P(M,T)$ in Eq. (10) suggests to compute its Laplace transform with respect to $T$
\[
\hat{P}(M,s) = \int_0^\infty P(M,T)e^{-sT}dT
\]
\[
= \lim_{\epsilon_1,\epsilon_2 \to 0} \frac{A_{N,B}}{M^{N+1}} \prod_{i=1}^{N-1} \int_0^\infty dq_i e^{-\frac{q_i^2}{2M}} e^{-\epsilon_1|q_i| - \epsilon_2|q_i|'} \times \Theta(q_i,\ldots,q_i') \frac{\Theta(q_1,\ldots,q_N)}{2M^2 + s} \times \Theta(q_1,\ldots,q_N) \frac{q_i'}{2M^2 + s},
\]
where we have regularized the integrals over $q_N$ and $q_i'$ with the help of this exponential term $e^{-\epsilon_1|q_i| - \epsilon_2|q_i'|}$. The next step is to expand the determinants in Eq. (C12) with respect to their last column. This yields
\[
\hat{P}(M,s) = \frac{4A_{N,B}}{M^{N+1}} \prod_{i=1}^{N-1} \int_0^\infty dq_i e^{-\frac{q_i^2}{2M}} \sum_{j=1}^N (-1)^{j+l} \times \det[M_{j,N}(\Theta_N)] \det[M_{i,N}(\Theta_N)] g_j(s) g_l(s),
\]
where
\[
g_j(s) = \lim_{\epsilon \to 0} \int_0^\infty dq \frac{q_j \cos\left(q - \frac{j\pi}{2}\right)}{q^2 + 2Ms} e^{-\epsilon|q|}.
\]
This integral (C14) can be easily evaluated using residues; this yields
\[
g_j(s) = \frac{\pi}{2} (M \sqrt{2s})^{-j-1} e^{-s^{\frac{1}{2}}}. \tag{C15}
\]
Using this result (C15) in the expression above (C13), one obtains
\[
\hat{P}(M,s) = \frac{\pi^2 A_{N,B}}{M^{N+2}} \prod_{i=1}^{N-1} \int_0^\infty dq_i e^{-\frac{q_i^2}{2M}} \sum_{j=1}^N (-1)^{j+l} \times \det[M_{j,N}(\Theta_N)] \det[M_{i,N}(\Theta_N)] (M \sqrt{2s})^{j-l-2} e^{-s^{\frac{1}{2}}}.
\]
Using the results shown above [Eqs. (C14) and (C15)] we can write $\hat{P}(M,s)$ as
\[
\hat{P}(M,s) = \frac{\pi^2 A_{N,B}}{M^{N+2}} \prod_{i=1}^{N-1} \int_0^\infty dq_i e^{-\frac{q_i^2}{2M}} \times \sum_{j=1}^N (-1)^{j+l} \det[M_{j,N}(\Theta_N)] \det[M_{i,N}(\Theta_N)] \times \int_0^\infty dq \frac{\sin\left(2q - (j + l)\frac{\pi}{2}\right)}{s + \frac{q^2}{2M^2}}.
\]
Under this form (C17) it is easy to invert the Laplace transform to obtain $P(M,T)$, and finally one obtains
\[
\int_0^1 P_{N,B}(M,\tau_M)d\tau_M = -\frac{\pi A_{N,B}}{M^N+1} \int_0^\infty dq e^{-\frac{q^2}{2M^2}} \sum_{j=1}^N (-1)^{j+l} q_j^{j+l-1} \times \det[M_{j,N}(\Theta_N)] \det[M_{i,N}(\Theta_N)] \sin\left(2q_N - (l + j)\frac{\pi}{2}\right).
\]
Finally, using the identity (C8) together with the explicit expressions (C7) and (C18), one obtains
\[
A_{N,B} = \frac{\pi}{N} a_{N,B} = \frac{2^{2N-N/2} N^{\frac{N}{2}+1}}{\pi^{N/2+1} \prod_{j=1}^N j!},
\]
as given in the text in Eq. (95).

2. Nonintersecting Brownian excursions

In this section we compute the constant $A_{N,E}$ of the formula (123) with the same method. Let us recall the joint distribution of $M$ and $\tau_M$ for the $N$ vicious excursions,
\[
P_{N,E}(M,\tau_M) = \frac{A_{N,E}}{M^{N+1}} \prod_{i=1}^{N-1} \left\{ \sum_{k=1}^\infty n_i^2 e^{-\frac{2s^2 n_i^2}{M^2}} \right\} \times \Omega_N(\{n_i\}|M,\tau_M) \Omega_N(\{n_i\}|M,1 - \tau_M),
\]
and where
\[
\Omega_N(\{n_i\}|M,1) = \sum_{k=1}^\infty (-1)^k k^2 \Delta_N(n_i^2, k)^2 e^{-\frac{2s^2}{M^2} k^2}, \tag{C21}
\]
with the same notation as in the main text $\{n_i\} = (n_1, \ldots, n_{N-1})$ and $\{n_i^2\} = (n_1^2, \ldots, n_{N-1}^2)$. The normalization of this joint PDF is
\[
\int_0^\infty dM \int_0^1 d\tau_M P_{N,E}(M,\tau_M) = 1. \tag{C22}
\]
As in the case of bridges we can use the result of Ref. [30] of the cumulative distribution of the maximum of $N$ vicious excursions,
\[
F_{N,E}(M) = \frac{a_{N,E}}{M^{2N+N}} \times \sum_{n_1} \cdots \sum_{n_N} \left\{ \prod_{i=1}^N n_i^2 \Delta_N^2(n_1^2, \ldots, n_N^2) e^{-\frac{2s^2}{M^2} n_i^2} \right\}, \tag{C23}
\]
where $\Delta_N^2$ is the square of the Vandermonde determinant,
\[
\Delta_N(n_1^2, \ldots, n_N^2) = \det_{1 \leq i, j \leq N} n_i^{2(j-1)}, \tag{C24}
\]
and $a_{N,E}$ is a normalization constant obtained by imposing $\lim_{M \to \infty} F_{N,E}(M) = 1$:
\[
a_{N,E} = \frac{\pi 2^{2N+N}}{2^{N^2/2+2} \prod_{j=0}^{N-1} \Gamma(2 + j)^\frac{1}{2} + j). \tag{C25}
\]
The derivative of the cumulative gives the PDF of the maximum

\[ P_{N,E}(M) = \frac{dF_{N,E}(M)}{dM} = \frac{a_{N,E}}{M^{2N^2+N+1}} \times \sum_{n_1} \cdots \sum_{n_N} \left\{ \left( -(2N^2 + N) + \frac{N\pi^2}{M^2}n_N^2 \right) \sum_{i=1}^N n_i^2 \right\} \times \sum_{i=1}^N n_i^2 \Delta_N^2(n_1^2, \ldots, n_N^2)e^{-\frac{\pi^2}{2M^2}n_N^2} \}.

(C26)

The term \( \frac{N\pi^2}{M^2}n_N^2 \) in the parentheses of the second line comes from the derivative of the exponential, giving a factor \( \frac{\pi^2}{M^2} \sum_i n_i^2 \) that can be simplified by the symmetry in the exchange of \( n_i \) and \( n_N \).

Now we have to compute the marginal \( P_{N,E}(M) \) by integrating the joint PDF given by Eq. (C20) over \( \tau_M \),

\[ P_{N,E}(M) = \int_0^1 d\tau_M P_{N,E}(M, \tau_M), \quad (C27) \]

to identify our constant \( A_{N,E} \). As in the Bridges case, we compute instead the function of \( T \),

\[ P(M,T) = \frac{A_{N,E}}{MN^{(2N+1)+3}} \sum_{n_1}^{\infty} \frac{n_1^2 e^{-\frac{x^2}{\sqrt{M}}}}{\Delta_N^2} \times \int_0^T d\tau_M \Omega_N((n_1)|M,\tau_M) \Omega_N((n_1)|M,T-\tau_M), \]

(C28)

and at the end we will recover

\[ P_{N,E}(M) = P(M,T = 1). \quad (C29) \]

The formula for \( P(M,T) \) is best treated with a Laplace transform, because the convolution in the second line of Eq. (C28) becomes a simple product. Taking the Laplace transform with respect to \( T \), one has

\[ \hat{P}(M,s) = \int_0^\infty dT e^{-sT} P(M,T) \]

\[ = \frac{A_{N,E}}{MN^{(2N+1)+3}} \sum_{n_1}^{\infty} \frac{n_1^2 e^{-\frac{x^2}{\sqrt{M}}}}{\Delta_N^2} \times \left\{ \hat{\Omega}_N((n_1)|M,s) \right\}^2, \]

(C30)

the Laplace transform of \( \Omega_N \) being

\[ \hat{\Omega}_N((n_1)|M,s) = \int_0^\infty dT e^{-sT} \Omega_N((n_1)|M,T) \]

\[ = \lim_{\epsilon \rightarrow 0} \sum_{k=1}^{\infty} (-\epsilon)^k \Delta_N(n_1^2, \ldots, n_N^2,k^2) \times \frac{2N^2}{M^2}k^2 \times \frac{2N^2}{M^2}k^2 \]

(C31)

This \( \epsilon < 1 \) has been introduced to regularize the integrand so that one can permute the integral from the Laplace transform and the infinite sum over \( k \). Writing the Vandermonde determinant as

\[ \Delta_N(n_1^2, \ldots, n_N^2,k^2) = \prod_{i=1}^{N-1} (k^2 - n_i^2) \times \prod_{1 \leq i < j \leq N-1} (n_j^2 - n_i^2), \quad (C32) \]

one can use the identity, shown below in Eq. (D1), with \( p^2 = 2M^2s/\pi^2 \) with \( s > 0 \) to get

\[ \hat{\Omega}_N((n_1)|M,s) = (-1)^N \sqrt{2\pi} M^{3/2} \prod_{i=1}^{N-1} \frac{n_i^2 + 2M^2s}{\pi^2} \times \prod_{1 \leq i < j \leq N-1} (n_j^2 - n_i^2). \]

(C33)

Under this form, we are ready to take the inverse Laplace transform of \( \hat{P}(M,s) \) using the Bromwich integral:

\[ P(M,T) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} ds e^{sT} \hat{P}(M,s), \]

(C34)

where \( \gamma \) is such that the vertical line keeps all the poles of the integrand to its left. The poles in the complex plane are located in \( s = -(k^2\pi^2)/(2M^2) \), for \( k = 1, 2, \ldots \), on the negative real axis, so that \( \gamma = 0 \) is well suited. We close the contour with a semicircle to the left. By the Jordan lemma, the integral on the semicircle when \( R \rightarrow \infty \) is zero. The residue theorem then gives

\[ P(M,T) = \frac{A_{N,E}}{MN^{(2N+1)+3}} \sum_{n_1}^{\infty} \frac{n_1^2 e^{-\frac{x^2}{\sqrt{M}}}}{\Delta_N^2} \times \left\{ \sum_{k=1}^{\infty} \frac{2k^2M^2}{\pi^2} e^{-\frac{\pi^2}{2M^2}k^2T} \times \prod_{1 \leq i < j \leq N-1} (n_j^2 - n_i^2) \times \left[ \frac{3}{2} \frac{2N^2}{M^2}k^2T \right] \right\}. \]

(C35)

Taking \( T = 1 \) and relabeling \( k = n_i \), one obtains

\[ P_{N,E}(M) = P(M,T = 1) = \frac{A_{N,E}}{MN^{(2N+1)+1}} \left( \frac{2}{\pi^2} \right) \times \sum_{n_1=1}^{N} \cdots \sum_{n_N=1}^{N} n_1^2 \Delta_N^2(n_1^2, \ldots, n_N^2)e^{-\frac{\pi^2}{2M^2}n_N^2} \times \left[ \frac{3}{2} \frac{2N^2}{M^2}n_i^2 \right] \]

(C36)

In order to compare this formula with Eq. (C26), we need to simplify the last term in the brackets. To this end, we introduce the function \( g(n) \equiv g(n_1, \ldots, n_N) \):

\[ g(n) = \prod_{i=1}^{N} n_i^2 \Delta_N^2(n_1^2, \ldots, n_N^2). \]

(C37)
which is totally symmetric under the exchange of any couple of its arguments. One has
\[
\sum_{n_1} \cdots \sum_{n_N} \left( \sum_{i=1}^{N-1} \frac{2n_i^2 N^2 - n_i^2}{n_N^2 - n_i^2} \right) g(n) \\
= \sum_{n_1} \cdots \sum_{n_N} \left( \sum_{i=1}^{N-1} \frac{2(n_i^2 - n_j^2) + 2n_i^2}{n_N^2 - n_i^2} \right) g(n) \\
= \sum_{n_1} \cdots \sum_{n_N} \left( 2(N - 1) + \sum_{i=1}^{N-1} \frac{2n_i^2}{n_N^2 - n_i^2} \right) g(n)
\]

where we change the role of \(n_i\) and \(n_N\) in the last line. Equating the last line with the first, one obtains
\[
\sum_{n_1} \cdots \sum_{n_N} \left( \sum_{i=1}^{N-1} \frac{2n_i^2 N^2 - n_i^2}{n_N^2 - n_i^2} \right) g(n) = \sum_{n_1} \cdots \sum_{n_N} (N - 1) g(n).
\]

Inserting this identity in formula (C36), the comparison with Eq. (C36) is straightforward, and one obtains
\[
A_{N,E} = N \pi^2 a_{N,E} = \frac{N \pi^{2N+2} \Gamma(2 + \frac{N}{2})}{2 \Gamma(N/2) \prod_{j=0}^{N-1} \Gamma(2 + j) \Gamma(\frac{3}{2} + j)},
\]

as given in the text in Eq. (120).

**APPENDIX D: IDENTITY**

We want to show the identity
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2 \prod_{i=1}^{N-1} (k^2 - n_i^2)}{p^2 + k^2} = \frac{(-1)^N \pi^2}{2 \sinh(\pi p)}.
\]

This is shown by induction on \(N\): we thus first analyze the case \(N = 1\) of this identity,
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2}{p^2 + k^2} = \frac{\pi}{2 \sinh(\pi p)},
\]

for any real \(p \neq 0\) (indeed, this has a limit for \(p = 0\) and one can extend the result for \(p = 0\)). The first step is to separate the series in the left-hand side into two parts:
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2}{p^2 + k^2} = \lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2 + p^2 - p^2}{p^2 + k^2} = \lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2 + p^2}{p^2 + k^2}.
\]

The first series is geometric and one has
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k = \lim_{\epsilon \to 1} \left( \frac{1}{1 + \epsilon} - 1 \right) = -\frac{1}{2}.
\]

In the second, one can permute the limit with the sum because of the normal convergence
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{p^2}{p^2 + k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + p^2} = \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{p^2}{k^2 + p^2} - \frac{1}{p^2}.
\]

To compute the right-hand side, we consider the function on the complex plane
\[
f(z) = \frac{p^2 e^{i\pi z}}{z^2 + p^2 e^{2i\pi z} - 1}.
\]

This function has simple poles in \(z = k\) with \(k \in \mathbb{Z}\) and in \(\pm ip\). One can verify that with \(z = Re^{i\theta}\),
\[
|zf(z)| \leq \frac{R^2 p^2}{2|z^2 + p^2| |\sinh(\pi R \sin \theta)|} \rightarrow R \to \infty 0.
\]

Then using the Jordan lemma, one has that the contour integral of \(f(z)\) along the infinite circle centered in \(z = 0\) is zero. Using the residue theorem, one finds
\[
0 = \sum_{k \in \mathbb{Z}} \left( 2i \pi \frac{i}{2\pi} \frac{p^2 e^{i\pi k}}{k^2 + p^2} \right) + (2i \pi) \frac{p^2 e^{-\pi p}}{2i p} - \frac{1}{2i p} e^{2\pi p} - 1.
\]

or
\[
\sum_{k \in \mathbb{Z}} (-1)^k \frac{p^2}{k^2 + p^2} = \frac{\pi p}{\sinh(\pi p)}.
\]

Inserting Eq. (D9) in Eq. (D5) and Eq. (D4) in Eq. (D3), one has
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2}{p^2 + k^2} = -\frac{1}{2} - \frac{1}{2} \frac{\pi p}{\sinh(\pi p)} - \frac{1}{2} = -\frac{\pi}{2} \sinh(\pi p).
\]

which is the announced result for \(N = 1\).

Let us prove the general result by induction. Consider the quantity
\[
\lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2}{p^2 + k^2} = \lim_{\epsilon \to 1} \sum_{k=1}^{\infty} (-\epsilon)^k \frac{k^2 + p^2 - (p^2 + n_N^2)}{p^2 + k^2} \prod_{j=1}^{N-1} (k^2 - n_i^2)
\]

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With the lemma (which we show below)

\[
\lim_{\epsilon \to 0} \sum_{k=1}^{\infty} (-\epsilon)^k k^{2q} = 0, \quad \forall \, q \in \mathbb{N}^*,
\]

and the induction hypothesis at rank \( N \) (D1) that we use in the second term of Eq. (D11), we have proved the formula at rank \( N + 1 \) which ends the induction proof of the identity (D1).

In order to prove the lemma (D12), we put \( \epsilon = e^{-x} \) with the real \( x > 0 \), and we shall take the limit \( x \to +0 \). Denoting for \( q = 0, 1, 2, \ldots \),

\[
g_q(x) = \sum_{k=1}^{\infty} (-1)^k e^{-kx} k^{2q},
\]

one sees that

\[
\frac{d^2}{dx^2} g_q(x) = g_{q+1}(x).
\]

Using the fact that \( g_0(x) \) is a simple geometric series, we apply this relation to compute \( g_1(x) \):

\[
g_1(x) = \frac{d^2}{dx^2} g_0(x)
= \frac{d}{dx} \left( \frac{1}{1 + e^{-x}} - 1 \right)
= \frac{e^{-x}(e^{-x} - 1)}{(1 + e^{-x})^2}.
\]

The domain of definition of \( g_1(x) \) is the positive real axis, and in this domain, \( g_1(x) \) coincides with the function defined over all the real line:

\[
x \mapsto \hat{g}_1(x) = \frac{e^{-x}(e^{-x} - 1)}{(1 + e^{-x})^2}.
\]

One sees easily that \( \hat{g}_1(x) \) is an odd function \( \hat{g}_1(-x) = -\hat{g}_1(x) \), which implies that \( \lim_{x \to 0} g_1(x) = \hat{g}_1(0) = 0 \), which shows the lemma for \( q = 1 \). To go further, the Taylor series of \( \hat{g}_1(x) \) must have only odd powers of \( x \). Because they coincide on \( x > 0 \), this is true for \( g_1(x) \):

\[
\forall \, x > 0, \quad g_1(x) = \sum_{p=0}^{\infty} a_{1,p} x^{2p+1}.
\]

Indeed, this odd power series expansion extends to all \( g_q(x) \) (with other coefficients) thanks to the relation (D14) because we differentiate twice to obtain \( g_{q+1} \) from \( g_q \)

\[
\forall \, x > 0, \quad g_q(x) = \sum_{p=0}^{\infty} a_{q,p} x^{2p+1},
\]

and this proves that

\[
\lim_{x \to 0} g_q(x) = 0,
\]

which, regarding the definition (D13), ends the proof of the lemma (D12).

**APPENDIX E: SOME USEFUL FORMULAS FOR HERMITE POLYNOMIALS**

For our purpose, the following integral representation of the Hermite polynomial \( H_n(z) \) is useful:

\[
H_n(z) = \frac{2^n}{\sqrt{\pi}} e^{z^2} \int_0^\infty dt e^{-t^2} t^n \cos(2zt - n\pi/2).
\]

The formula used in the text is explicitly

\[
\int_0^\infty dq \, q^n \cos \left( q - \frac{n\pi}{2} \right) e^{-\frac{1}{2}q^2} = \sqrt{\pi} \left( \frac{M}{\sqrt{2t}} \right)^{n+1} e^{-\frac{n^2}{2}H_n \left( \frac{M}{\sqrt{2t}} \right)}.
\]

For the stars configurations we use the identity

\[
\int_0^\infty dq \, q^n \cos \left( q - \frac{n\pi}{2} \right) e^{-\frac{1}{2}q^2} = \sqrt{\pi} \left( \frac{M}{\sqrt{2t}} \right)^{n+1} e^{-\frac{n^2}{2}H_n \left( \frac{M}{\sqrt{2t}} \right)}
\]

(in particular with \( t = 1 - \tau_M \) and \( n = 1 \)).
