Universal Nonstationary Dynamics at the Depinning Transition

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We study the nonstationary dynamics of an elastic interface in a disordered medium at the depinning transition. We compute the two-time response and correlation functions, found to be universal and characterized by two independent critical exponents. We find a good agreement between two-loop functional renormalization group calculations and molecular dynamics simulations for the scaling forms, and for the response aging exponent \( \theta_R \). We also describe a dynamical dimensional crossover, observed at long times in the relaxation of a finite system. Our results are relevant for the nonsteady driven dynamics of domain walls in ferromagnetic films and contact lines in wetting.

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The universal glassy properties that emerge from the frustrating competition between elasticity and disorder are relevant for many experimental systems, such as interfaces describing magnetic [1–4] and ferroelectric [5,6] domain walls, contact lines of fluids [7,8], and fracture [9,10]. Disorder leads to pinning, affecting in a dramatic way their dynamical properties. In particular, when driven by an external force \( f \) at zero temperature, disorder leads to a depinning transition at a threshold value \( f = f_c \), below which the interface is immobile, and above which steady-state motion sets in. For \( f \approx f_c \) it has been fruitful to regard the depinning transition as a critical phenomenon, with the mean velocity \( v \) as an order parameter, \( v \sim (f - f_c)^\beta \), and with a characteristic length \( \xi \) playing the role of the divergent correlation length \( \xi \sim (f - f_c)^{-\nu} \), \( \beta \) and \( \nu \) being universal exponents [11]. Near the critical point, however, the time needed to reach such a nonequilibrium steady state can be very long, since the memory of the initial condition persists for length scales larger than a growing correlation length \( \ell(t) \sim t^{1/\zeta} \), with \( \zeta \) the dynamical exponent [12,13]. Being only limited by the divergent steady correlation length \( \xi \) or the system size \( L \), the resulting nonsteady critical regime is macroscopically large, \( t \lesssim \xi^2, L^2 \). It is thus relevant for experimental protocols. Analogously to nondriven systems relaxing to their critical equilibrium states [14,15], we show here that the transient dynamics of a driven disordered system displays interesting, though different, universal features.

Dynamical properties are characterized by two-time \( t, t_w \) response and correlation functions, describing the time evolution of the system as a function of its “age” or waiting time \( t_w \) from a given initial condition at \( t = 0 \). Progress was achieved in the understanding of the steady state, where these functions depend only on \( t - t_w \). Functional renormalization group (FRG) calculations [16–19] allowed one to compute the critical exponents describing various universality classes, and powerful algorithms were developed to elucidate the low temperature dynamical phase diagram [20]. In contrast, little is known about the more difficult transient regime, where the time-translation invariance is broken. Yet the first steps in that direction have unveiled rich and universal behaviors, including slow dynamics [13] and aging properties characterized by new exponents [12].

Here we focus on elastic interfaces of dimension \( d \) (\( d = 1 \) for an elastic line) parametrized by a scalar field \( u_{x,t} \) describing their position in a \( (d + 1) \)-dimensional disordered medium. The driven overdamped dynamics of this model system obeys the equation of motion

\[
\eta \partial_t u_{x,t} = c \nabla^2 u_{x,t} + F(x, u_{x,t}) + f,
\]

where \( \eta \) is the friction coefficient, \( c \) the elastic constant, and \( F(x, u) \) a quenched random pinning force with disorder averaged correlations \( \langle F(x, u)F(x', u') \rangle = \Delta(u - u') \delta^d(x - x') \). Under an applied external force \( f \), the velocity is \( v = \frac{L^{-1}}{t} \int d^d x \partial_t u_{x,t} \). In this Letter we consider a flat initial configuration \( u_{x,0} = 0 \) but our results hold for any short ranged correlated initial conditions. Denoting \( \hat{u}_{q,t} \) the spatial Fourier transform of \( u_{x,t} \), we focus on the linear response \( \mathcal{R}_{q,t}^{u} \) to a small external field \( \hat{h}_{-qt} \) and the correlation function \( \mathcal{C}_{q,t}^{u} \):

\[
\mathcal{R}_{q,t}^{u} = \frac{\delta \hat{u}_{q,t}/\delta \hat{h}_{-qt}}{\delta \hat{h}_{-qt}}, \quad \mathcal{C}_{q,t}^{u} = \hat{u}_{q,t} \hat{u}_{-q,t}.
\]

The main result of this Letter is to establish, both via numerical calculations and additional analytical work, that these central observables take the scaling form

\[
\mathcal{R}_{q,t}^{u} = \left( t/t_w \right)^{\theta_R} q^{-2} F_R[q^2(t - t_w), t/t_w],
\]

\[
\mathcal{C}_{q,t}^{u} = q^{-(1 + 2\theta_C)(t/t_w)^{\theta_C - 1}} F_C[q^2(t - t_w), t/t_w],
\]

with \( \theta_R \) and \( \theta_C \) two new universal critical exponents, i.e., independent of the usual depinning exponents. These are defined such that \( F_{R,C}(y_1, y_2 \to \infty) \sim f_{R,C}(y_1) \) for fixed \( y_1 \).
In Eqs. (3) and (4), \(F_{R,C}\) are universal scaling functions (up to a nonuniversal amplitude) and \(\zeta\) the roughness exponent. In the limit \(y_1 \to 0\), \(y_2\) fixed, one finds \(F_{R}(y_1, y_2) \sim y_1^{(2-\zeta)/2} g_{R}(y_2)\), \(F_{C}(y_1, y_2) \sim y_1^{(2+\zeta)/2} g_{C}(y_2)\), i.e., a well-defined \(q\) \(\to 0\) limit. These scaling forms were predicted in Ref. [12] based on a one-loop FRG calculation. One may question, however, whether this lowest order in the \(d = 4 - \epsilon\) dimensional expansion is accurate enough to describe interfaces of experimental interest \(d = 1, 2\). In addition, no prediction for \(\theta_C\) was obtained. Here we firmly establish that the above scaling forms hold and we provide a reliable determination of \(\theta_R\) and \(\theta_C\) in \(d = 1\). We also perform a two-loop FRG calculation, as is known to be required for a consistent theory of depinning [19].

Most of the numerical studies of the transient dynamics have focused so far on one-time quantities which can be obtained from \(C_{n}^{\theta}\) and \(\mathcal{R}_{q}^{2}\) in Eqs. (3) and (4). The structure factor \(S_{q}(t) \equiv C_{n}^{\theta}\) was found [13] to behave as

\[
S_{q}(t) \equiv C_{n}^{\theta} \sim q^{-d(2+\zeta)/2} F[q(t)], \quad \ell(t) \to t^{1/\zeta},
\]

where \(F(y) \sim c^{d}\), a constant, for \(y > 1\) and \(F(y) \sim y^{d+\zeta}/\zeta\) for \(y \ll 1\). The relaxational dynamics is thus dictated by a single growing length, separating the small, steady-state equilibrated scales, from the large ones retaining a long-time memory of the initial condition. Equation (5) is obtained from (4) in the limit \(t \to t_w\) (i.e., \(y_2 \to 1\), \(y_1 \to 0\)) with \(y^*\) [i.e., \(y_1 y_2/(y_2 - 1)\)] fixed. The analogy with standard critical phenomena suggests for the velocity, the scaling form,

\[
v(t, f) = b^{-\beta/\nu} G[b^{-\nu} t, b^{1/\nu}(f - f_c), b^{-1} L]
\]

where \(b\) is an arbitrary rescaling factor, numerically verified in Ref. [13]. For \(f = f_c\) and \(t \ll L^{\nu}/\nu\) it implies that \(v(t) \sim t^{-\beta/\nu}\) and also that

\[
dv(t)/dt \propto A t^{(2-\zeta)/2},
\]

where we used the exact relations \(\beta = \nu(1 - \zeta)\) and \(\nu = 1/(2 - \zeta)\), from statistical tilt symmetry (STS) [16]. We now check that this scaling of one-time observables for \(f = f_c\) is consistent with the two-time scaling Eq. (4). Indeed Eq. (6) results by combining the exact relation [18]

\[
\frac{d}{dt} v(t) = \int_{0}^{t_{w}} ds \partial_{t} \mathcal{R}_{q}^{2} - \frac{\partial_{t}}{L}
\]

and the limit \(q \to 0\) of Eq. (3)

\[
\mathcal{R}_{q}^{2} \sim (t - t_{w})^{(2-\zeta)/2} (L/t_w)^{\theta_{q}} g_{R}(t/t_{w}),
\]

with \(g_{R}(x) \sim c^{d}\) a constant for \(x \gg 1\).

Let us now focus on two-time quantities. To check Eqs. (3) and (4) we have performed numerical simulations of Eq. (1) in the case of elastic lines, \(d = 1\), experimentally relevant for many two dimensional systems, e.g., films. To study the nonstationary dynamics at \(f = f_c\) we discretize Eq. (1) in the \(x\) direction, \(u_{i,x} = u_{i}(t)\), with \(i = 0, \ldots, L - 1\), and use the method described in Ref. [13]. We start at \(t = 0\) with a flat configuration, \(u_{i}(t = 0) = 0\), and monitor correlation and response functions at the exact sample critical force \(f_c\) [21].Numerically, it is more convenient to work with the local integrated response \(\rho_{n}^{\alpha} = \int_{0}^{t_{w}} ds \partial_{t} R_{Q}^{2} - \frac{\partial_{t}}{L}
\]

where \(\rho_{n}^{\alpha}\) denotes the integral over the first Brillouin zone and zero-mode integrated response \(\rho_{n}^{\alpha} = \int_{0}^{t_{w}} ds \partial_{t} R_{Q}^{2} - \frac{\partial_{t}}{L}
\]. From Eq. (3) we predict,

\[
\rho_{n}^{\alpha}(t_{w}) = \frac{\rho_{n}^{\alpha}(t_{w})}{(t_{w} - t_{w})^{1/\zeta}} = \hbar(t_{w})
\]

where both \(\hbar(y)\) and \(\hbar(y)\) behave as \(y^{-1+\theta_{q}}\) for \(y \to \infty\). To implement the local (zero-mode) response we define the observable \(w_{i}(t) = u_{i}(t) - u_{c}(t)\) \(w_{i}(t) = u_{i}(t)\) where \(u_{c}(t) = \frac{1}{L} \sum_{i} u_{i}(t)\) and then compute,

\[
\rho_{n}^{\alpha}(q) = \lim_{a \to 0} \frac{1}{L} \sum_{i} [w_{i}(t) - w_{i}(t)] \sigma_{s} \alpha^{-1},
\]

where \(w_{i}(t)\) is the solution of Eq. (1) with \(u_{i}(t) = 0\) and an additional perturbative force \(\sigma_{s} \alpha(t - t_{w})\). We take random numbers \(\sigma_{s} = \pm 1\) uncorrelated from site to site for computing \(\rho_{n}^{\alpha}\) and \(\sigma_{s}^* = \sigma_{s}^*\) for computing \(\rho_{n}^{\alpha}\). The value of \(\alpha\) is chosen small enough to guarantee linear response [22]. In Fig. 1 we show the numerical results for \(\rho_{n}^{\alpha}\) and \(\rho_{n}^{\alpha}\), for \(L = 2048\) averaged over 10,000 disorder realizations. We see that the predicted scaling forms, Eq. (8), describe well the data. For \(t/t_{w} \gg 1\) we observe a well-developed power-law behavior with an aging exponent \(\theta_{R} = -0.6 \pm 0.05\) which is indistinguishable for both responses, \(\rho_{n}^{\alpha}(t - t_{w})^{1/\zeta} - \rho_{n}^{\alpha}(t - t_{w})^{-\zeta/2} \sim (t/t_{w})^{1+\theta_{q}}\), as predicted. How does this numerical estimate for \(\theta_{R}\) compare with the previous FRG approach of Ref. [12]? The one-loop result for \(\theta_{R} = -\xi + O(\epsilon^2)\), setting \(\epsilon = 3\) gives \(\theta_{R} = -1/3\). Incidentally, up to one-loop order, one finds the relation \(\theta_{R} = (z - 2)/\zeta\), which, using the numerical estimate \(z = 1.5\) [13], yields again \(\theta_{R} = -1/3\). Although it goes in the
right direction, it is still far from our numerical result. To see whether the FRG predictions can be improved we have computed $\mathcal{R}^{q}_{tt'}$ up to two-loop order [23]. The starting point is Eq. (1). Response and correlations are then obtained from the standard dynamical (disorder averaged) Martin-Siggia-Rose action $S$ which reads here

$$S = \int_{t>t'} \int_q i\tilde{u}_q [(q^2 + \delta t)\delta_{tt'} + \tilde{\Sigma}_{tt'}]u_{-q'} - \frac{1}{2} \int_{t>t'} \int_{x,x'} i\tilde{u}_x i\tilde{u}_{x'} \Delta(u_{xt} - u_{x't}), \tag{10}$$

where $\Delta(u)$ is the force-force correlator and $\tilde{\Sigma}_{tt'}$ is the self-energy. As a result of the covariance of the action under STS [16] the self-energy has the structure $\tilde{\Sigma}_{tt'} = \Sigma_{tt'} - \delta_{tt'} \int_0^t dt \tilde{\Sigma}_{tt}$. It was computed to one loop in Ref. [12] and at two-loop order the perturbation theory leads to diagrams similar to the one contributing the dynamical exponent $z$, as depicted in Fig. 10 of Ref. [19], with the constraint that here, the time variables are positive. The (bare) response function $\mathcal{R}_{tt'}^q = \langle i\tilde{u}_q u_{-q'} \rangle_S$ is then computed from the exact identity

$$\mathcal{R}_{tt'}^q = R_{tt'}^q - \int_{t'<t_1<t_1<\cdots<t_1} R_{tt_1}^q \Sigma_{tt_1}^q R_{t_1t'}^q + \int_{t_1<t_1<\cdots<t_1} R_{tt_1}^q \Sigma_{t_1t'_1}^q \left[ \int_{0<t_2<t_1} \Sigma_{0t_2}^q \right]. \tag{11}$$

where $R_{tt'}^q = \theta(t-t') e^{-q^2(t-t')} \varphi$ is the response in the absence of disorder.

Reexpressing in terms of $\Delta(u)$, corrected to the same order, and using the FRG fixed point equation, one explicitly shows that it has the scaling form as in Eq. (7). We then find that no new independent divergence occurs in $t/t'$ order and, hence, that, to two-loop order, the relation $\theta_R = (z - 2)/z + O(\varepsilon^3)$ continues to hold. Our numerical result, however, indicates that this relation cannot hold to all orders in $\varepsilon$, i.e., one must have $\theta_R \neq (z - 2)/z$, implying that $\theta_R$ is indeed a new independent exponent. One way to understand the FRG result is then to rewrite more explicitly,

$$\theta_R = -\frac{\varepsilon}{9} + \left( \frac{1}{162\gamma\sqrt{2}} - \frac{\log 2}{108} - \frac{23}{648} \right) \varepsilon^2 + O(\varepsilon^3)$$

$$= -0.11111\ldots \varepsilon - 0.03395\ldots \varepsilon^2 + O(\varepsilon^3), \tag{12}$$

with $\gamma = 0.54822\ldots$. We note that if we set $\varepsilon = 3$ in that expression (12) assuming the $O(\varepsilon^3)$ to be small, we obtain $\theta_R = -0.64\ldots$, very close to the numerical value. Hence we conclude that although corrections to 3 loop and higher to $\theta_R - (z - 2)/z$ must be large, in $\theta_R$ they must be small. This provides one way of interpreting our results, and motivation for future analytical work.

We have also checked numerically the scaling form for the correlation function in Eq. (4). It is more convenient to compute the autocorrelation function $C_{tt'}^{u^0} = \mathbb{P}_{x,t} u_{x,t}$ ob-
function is given exactly by \( R_{u_c} = \theta(t - t_w) \nu(u_t, f)/\nu(u_c, f) \). In the usual model \( \eta \nu(u_t, f) = F(u_t) + f \), near the threshold, \( \delta f = f - f_c \ll f_c \), the particle spends most time near the zero force point (set to be at \( u = 0 \)). For a smooth force field we can write \( \eta \nu = \gamma u_i^2 + \delta f \) which yields \( \nu(t) \sim t^{-2} \) and

\[
R_{u_c} = \frac{\sin^2(t + \sqrt{\delta f}/\eta)}{\sin^2(t - \sqrt{\delta f}/\eta)} = \frac{\sin^2(\sqrt{\delta f}/\eta)}{(t/t_w)^{\gamma}}
\]

for \( \delta f \to 0 \). Hence \( \theta_R(d = 0) = -2 \), consistent with a monotonic dependence of \( \theta_R \) with \( d \), and \( \beta(d = 0) = 1/2 \).

Next, we confirm that the results of the \( d = 0 \) model are relevant for the interface for \( \ell(t) > L \). As shown in Fig. 3(d), we checked that for interfaces of sizes \( L = 32 \) and small \( \delta f \ll f_c \), the reparametrized velocity has a nice parabolic shape near the zero force point, \( \nu(u, f) = \gamma(u + a\delta f)^2 + b\delta f \) where the constants \( b, \gamma > 0 \) (their size dependence will be studied elsewhere [23]). A being found irrelevant, this result is consistent with the steady-state value \( \beta = 1/2 \) found for the interface in the regime \( \delta f^{-\nu} \gg L \) [28], and predicts a crossover to an effective \( \theta^{\nu}_{\ell} = -2 \) in the fixed \( L, \ell \) limit for the interface.

For modeling contact lines [7] the term \( \nabla^2 u_{x,z} \) in Eq. (1) is replaced by a long range elastic force \( \int [u(x', t) - u(x, t)]/[x - x']^2 \). In this case, using the same numerical method, we confirm the scaling forms (3) (replacing \( q^{-2} \to q^{-1} \)) and measure \( \theta_C = -1.2 \pm 0.1 \) and \( \theta_R = -0.5 \pm 0.1 \). The same analysis leading to (12) gives \( \theta_R = -0.22 \) to one loop and \( \theta_R = -0.38 \) to two loop.

To conclude we have confirmed numerically the scaling forms for nonstationary dynamics at depinning, for model systems of experimental relevance. The exponent \( \theta_R \) is found in reasonable agreement with FRG predictions. An interesting dimensional crossover was found at large time. We hope this motivates new experiments, e.g., in magnets and wetting.

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[22] Details will be published elsewhere.