Integrability in refined topological strings

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Plan

- Refined topological strings on toric Calabi-Yau threefolds.
- Ding-Iohara-Miki algebra is the symmetry algebra of topological strings.
- R-matrix of the Ding-Iohara-Miki algebra and geometric transitions on CY.

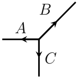
Topological strings

Topological strings on toric CY_3 compute

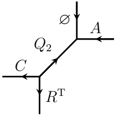
- effective action of the $5d$ gauge theory ($=$ Nekrasov function)
- GW and DT invariant of CY_3
- q-W_N algebra conformal blocks
- (q, t)-matrix models

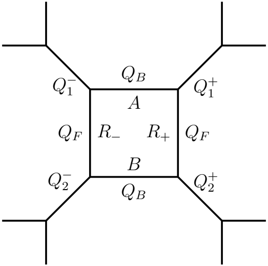
Toric CY and toric graphs

- Toric CY is a $\mathbb{T}^3$ fibration with base real $3d$ base. $\mathcal{B}$. Toric diagram $\mathcal{T} \subset \mathcal{B}$ — degeneration locus $\mathbb{T}^3 \to S^1$.
- Threefold examples:

1) $\mathbb{C}^3$: 

,

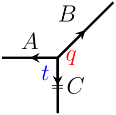
2) $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$: 

3) 

- The diagram is balanced: the integer valued tension vectors are conserved.

(Refined) topological vertex

To each trivalent vertex one associates an expression — topological vertex



$$= C_{ABC}(t,q) = q^{\frac{||B||^2+||C||^2}{2}} t^{-\frac{||B^{\mathrm{T}}||^2+||C^{\mathrm{T}}||^2}{2}} M_C^{(q,t)}\left(t^{-\rho}\right) \times$$

$$\sum_D \left(\frac{q}{t}\right)^{\frac{|D|+|A|-|B|}{2}} \chi_{A^{\mathrm{T}}/D}\left(q^{-C} t^{-\rho}\right) \chi_{B/D}\left(t^{-C^{\mathrm{T}}} q^{-\rho}\right)$$

Topological string amplitude is computed by the same recipe, as the Feynman rules, but with discrete sums over Young diagrams from the edges

$$Z_{\mathrm{top}} = \sum_{A_e} \prod_{\mathrm{edges}} Q^{|A_e|} \times \prod_{\mathrm{vertices}} C_{A_{e_1} A_{e_2} A_{e_3}}$$

Sum over diagrams is equivalent to a matrix model. Each balanced toric graph gives a network matrix model.

Topological strings from DIM algebra

Ding-Iohara-Miki (a.k.a. quantum toroidal, elliptic Hall, $U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_1)$) algebra:

$$G^{\mp}(z/w)\, x^{\pm}(z)\, x^{\pm}(w) \;=\; G^{\pm}(z/w)\, x^{\pm}(w)\, x^{\pm}(z)$$

$$+\,\text{ten more relations}$$

where $G^{\pm}(z) = \left(1 - q^{\pm 1} z\right)\left(1 - t^{\mp 1} z\right)\left(1 - (t/q)^{\pm 1} z\right)$.

Ginzburg, Kapranov, Vasserot '95, Ding, Iohara '96, Miki '98

DIM acts on the legs of the toric diagram. The vertices are intertwiners of the algebra: *Awata, Feigin, Shiraishi '11*

$$\Phi : \mathcal{F}_u^{(0,1)} \otimes \mathcal{F}_v^{(1,0)} \to \mathcal{F}_{-uv}^{(1,1)} \qquad C_{ABC} = \langle C|\Phi|A \otimes B\rangle$$

Commutativity with the intertwiners implies:

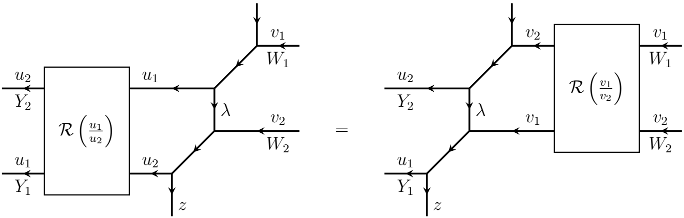
- W_N-constraints for q-CFT correlators,
- Loop equations and quantum spectral curves for (q,t)-matrix models,
- qq-character regularity for gauge theories.

DIM algebra has the universal R-matrix, following from the nontrivial coproduct:

$$\Delta^{\mathrm{op}} = R\Delta R^{-1}$$

If we choose two particular representations, R-matrix permutes them:

$$R : \mathcal{F}_u^{(1,0)} \otimes \mathcal{F}_v^{(1,0)} \to \mathcal{F}_v^{(1,0)} \otimes \mathcal{F}_u^{(1,0)}$$

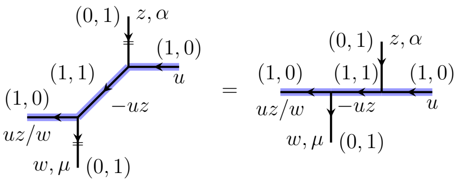


This R-matrix is the q-deformation of the instanton R-matrix *[Smirnov '13, Okounkov, Smirnov '16]*. We were able to compute the first three levels of this R-matrix explicitly using generalized Macdonald polynomials.

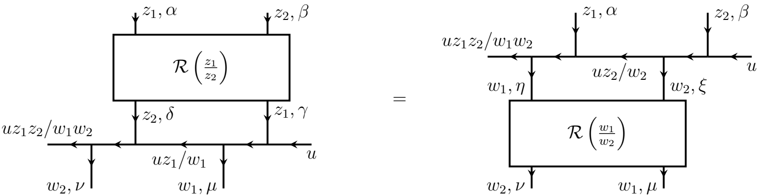
RTT relations

The $\mathcal{T}$-operator is built from topological vertices:

$\mathcal{T}^{\alpha}_{\beta}(u|z,w) =$



The "vertical" RTT relations:



Notice that the spectral parameters are different on the upper strands and on the lower ones. Thus, we have actually a generalization of the RTT-relations where the $\mathcal{T}$-operator changes the spectral parameters.

Anomaly

In the vertical basis (the basis of generalized Macdonald polynomials) the R-matrix is *diagonal* (times a permutation):

$$\mathcal{R}^{\alpha\beta}_{\gamma\delta}(x) = \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} R_{\alpha\beta}(x)$$

where $R_{\alpha\beta}(x)$ is an explicit rational function.
The RTT relations actually contain an *anomaly* — an extra scalar function $E_{q,t}$:

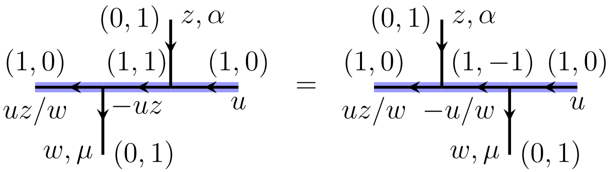
$$\mathcal{T}^{\beta}_{\nu}(u|z_2, w_2)\mathcal{T}^{\alpha}_{\mu}\left(\frac{uz_2}{w_2}\middle| z_1, w_1\right) =$$

$$= \mathcal{T}^{\alpha}_{\mu}(u|v_1, u_1)\mathcal{T}^{\beta}_{\nu}\left(\frac{uz_1}{w_1}\middle| z_2, w_2\right)\frac{R_{\mu\nu}\left(\frac{w_1}{w_2}\right)}{R_{\alpha\beta}\left(\frac{z_1}{z_2}\right)} E_{q,t}(z_1, w_1, z_2, w_2)$$

$E_{q,t}$ cannot be absorbed into the ration of two R-matrices, hoever it does not spoil the Yang-Baxter equation. The anomaly arises because we consider a "doubly infinite" algebra. It is related to the braiding matrices of the q-conformal blocks.

The geometric transition and integrals of motion

The action of *R*-matrix reminds of the "flop" transition in resolved conifold:



Despite the anomaly there are still integrals of motion:

$$\operatorname{tr} \mathcal{T}_1 \operatorname{tr} \mathcal{T}_2 = \operatorname{tr} \mathcal{T}_2 \operatorname{tr} \mathcal{T}_1$$

They coincide with the integrals of the ILW hierarchy *[Litvinov '13]*. The DIM *R*-matrix can also be understood as the *q*-deformation of the Liouville (or, more generally Toda) reflection operator. Fusion may be used to obtain the *R*-matrix for more sophisticated representations.

Conclusions and prospects

- Topological strings contain a new and interesting integrable structure.
- The key object is the DIM R-matrix, which can be explicitly computed.
- This result has many implications for q-CFT, matrix models and gauge theories. We hope to investigate them in the future.

Thank you for your attention!