STOCHASTIC LÉVY DIFFERENTIAL OPERATORS AND GAUGE FIELDS

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- The original Lévy Laplacian was defined for functions on the space $L_2(0,1)$ in the twenties of the last century. The value of the Lévy Laplacian on a function can be determined as an integral functional generated by the special form of the second order derivative or as the Cèsaro mean of second-order directional derivatives along vectors of some orthonormal basis $\{e_n\}$ in $L_2(0,1)$:

$$\Delta_L f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d_{e_k} d_{e_k} f(x)$$

- Arefieva and Volovich (1981) have considered the connection between the Yang-Mills fields and the divergence related to the Lévy Laplacian defined as an integral functional. The Yang-Mills fields has been considered as infinite dimensional chiral fields.

- Accardi, Gibilisco, Volovich (1994) have proved that the connection in the trivial bundle over an Euclidean space satisfies the Yang-Mills equations if and only if the corresponding parallel transport is a harmonic functional for the Lévy Laplacian defined as an integral functional.
- Volovich and Leandre have defined the stochastic analogue of that Laplacian (2002) and have shown that the similar theorem for the stochastic parallel transport and the Yang-Mills equations on a compact Riemannian manifold holds.

- In the deterministic case the solution of the Yang-Mills equations can been described by the Lévy Laplacian defined as the Cèsaro mean of second-order directional derivatives. We consider the stochastic analogue of this Laplacian and show that, unlike the deterministic case, the equivalence of the Yang-Mills equations and the Lévy-Laplace equation is not valid for such Laplacian. An equation equivalent to the Yang-Mills equations is obtained. This equation contains the stochastic Lévy divergence.

The Yang-Mills equations

A connection in the trivial vector bundle with base $\mathbb{R}^d$, fiber $\mathbb{C}^N$ and structure group $U(N)$ is defined as $u(N)$-valued C^∞-smooth 1-form $A(x) = A_\mu(x)dx^\mu$ on $\mathbb{R}^d$. If $\phi \in C^1(\mathbb{R}^d, u(N))$, then the covariant derivative ϕ in the direction of the vector field $(\frac{\partial}{\partial_\mu})$ is defined by the formula $\nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]$. The curvature corresponding to the connection A is the 2-form $F(x) = \sum_{\mu<\nu} F_{\mu\nu}(x)dx^\mu \wedge dx^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The Yang-Mills action functional has a form

$$-\int_{\mathbb{R}^d} tr(F_{\mu\nu}(x)F^{\mu\nu}(x))dx \tag{1}$$

The Yang-Mills equations are the Euler-Lagrange equations for (1) and have a form:

$$\nabla^\mu F_{\mu\nu} = 0. \tag{2}$$

These equations are equations on a connection A.

Parallel transport

Let

$$H := W_0^{2,1}([0,1],\mathbb{R}^d) =$$
$$= \{\gamma \text{ is absolutely continuous}, \ \gamma(0) = 0, \ \dot\gamma \in L_2((0,1),\mathbb{R}^d)\}$$

For any $\gamma \in H$ and $x \in \mathbb{R}^d$ we consider the equation

$$U_t^x(\gamma) = I_N - \int_0^t A_\mu(x+\gamma(s))U_s^x(\gamma)\dot\gamma^\mu(s)ds \tag{3}$$

The operator $U_1^x(\gamma)$ is parallel transport along the curve γ. We consider the function $H \ni \gamma \to U_1^x(\gamma)$.

Let $\{p_\mu\}$ be orthonormal basis in $\mathbb{R}^d$.
Let $h_n(t) = \sqrt{2}\sin n\pi t$.

Definition

The Levy Laplacian Δ_L is a linear mapping from $dom\Delta_L$ to the space of $M_N(\mathbb{C})$-valued functions on H defined by

$$\Delta_L F(\gamma) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} d_{p_\mu h_k} d_{p_\mu h_k} F(\gamma), \tag{4}$$

where $dom\Delta_L$ consists of all two times differentiable functions on H for which the right-hand side of (4) exists for all $\gamma \in H$.

Theorem

The following equality holds

$$\Delta_L U_1^x(\gamma) = -U_1^x(\gamma) \int_0^1 U_t^x(\gamma)^{-1} \nabla^\mu F_{\mu\nu}(x+\gamma(t)) U_t^x(\gamma) \dot{\gamma}^\nu(t) dt$$

Corollary

The following two assertions are equivalent:

1. *a connection A satisfies the Yang-Mills equations:* $\nabla_\mu F^\mu_\nu = 0$,
2. $\Delta_L U_1^x = 0$ *for some* $x \in \mathbb{R}^d$.

This fact has been proved by Accardi, Gibilisco, Volovich for the Lévy Laplacian defined as an integral functional.

Stochastic parallel transport

We denote the d-dimensional Brownian motion by $b_t = (b_t^1, \ldots, b_t^d)$ and the associated Wiener measure by P. The Itô differentials and Stratonovich differentials are denoted by db and by $\circ db$ respectively.

The stochastic parallel transport $U^x(b,t)$ $(x \in \mathbb{R}^d)$ associated with the connection A is a solution to the differential equation (in the sense of Stratonovich):

$$U^x(b,t) = I_N - \int_0^t A_\mu(x + b_s) U^x(b,s) \circ db_s^\mu \tag{5}$$

Sobolev spaces over the Wiener measure

Let $\mathfrak{H}$ be a real or complex Hilbert space. By the symbol $\|\cdot\|_p$ we denote the norm of $L_p(P, \mathfrak{H})$. The Sobolev norm $\|\cdot\|_{p,r}$ on the space $\mathcal{F}C^\infty(\mathfrak{H})$ of $\mathfrak{H}$-valued C^∞-smooth cylindrical function with compact support on $C_0([0,1], \mathbb{R}^d)$ is defined by

$$\|f\|_{p,r} = \sum_{k=1}^{r} (E(\sum_{i_1 \dots i_k = 1}^{\infty} \|\partial_{g_{i_1}} \dots \partial_{g_{i_k}} f\|_{\mathfrak{H}}^2)^{p/2})^{1/p},$$

where $\{g_n\}$ is an arbitrary orthonormal basis in H. The Sobolev space $W_r^p(P, \mathfrak{H})$ is the completion of $\mathcal{F}C^\infty(\mathfrak{H})$ with respect to the norm $\|\cdot\|_{p,r}$.

Stochastic Lévy Laplacian

The stochastic Lévy Laplacian Δ_L is a linear mapping from $dom\Delta_L$ to $L_2(P, M_N(\mathbb{C}))$ defined as

$$\Delta_L f(b) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \partial_{p_\mu h_k} \partial_{p_\mu h_k} f(b), \qquad (6)$$

where the sequence converges in $L_2(P, M_N(\mathbb{C}))$ and $dom\Delta_L$ consists of all $f \in W_2^2(P, M_N(\mathbb{C}))$ for which the right-hand side of (6) exists.

Theorem

The following equality holds

$$\Delta_L U^x(b,1) =$$

$$= U^x(b,1)(\int_0^1 U^x(b,t)^{-1} F_{\mu\nu}(x+b_t) F^{\mu\nu}(x+b_t) U^x(b,t)dt -$$

$$- \int_0^1 U^x(b,t)^{-1} \nabla^\mu F_{\mu\nu}(x+b_t) U^x(b,t) db_t^\nu)$$

Corollary

If the connection A satisfies the Yang-Mills equations $\nabla^{\mu} F_{\mu\nu} = 0$, *then*

$$\Delta_L U^x(b,1) =$$
$$= U^x(b,1) \int_0^1 U^x(b,t)^{-1} F_{\mu\nu}(x+b_t) F^{\mu\nu}(x+b_t) U^x(b,t) dt.$$

Stochastic Lévy Divergence

The stochastic Lévy divergence div_L is a linear mapping from $dom\, div_L$ to $L_2(P, M_N(\mathbb{C}))$ defined by the formula

$$div_L B(b) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \partial_{p_\mu h_k} B(b)(p_\mu h_k), \tag{7}$$

where the sequence converges in $L_2(P, M_N(\mathbb{C}))$ and $dom\, div_L$ consists of all

$$B \in L(H, W_2^1(P, M_N(\mathbb{C})))$$

for which the right-hand side of (7) exists.

Let $B^{A,x} \in L(H, W_2^1(P, M_N(\mathbb{C})))$ be defined by the formula

$$B^{A,x}(b)u = U^x(b,1)^{-1}\partial_u U^x(b,1)$$

Theorem

The following equality holds

$$div_L B^x(b) = -\int_0^1 U^x(b,s)^{-1}\nabla^\mu F_{\mu\nu}(x+b_s)U^x(b,s)db_s^\nu.$$

Theorem

The following two assertions are equivalent:

1. *a connection A satisfies the Yang-Mills equations:* $\nabla_\mu F^\mu_\nu = 0$*,*
2. $div_L B^{A,x} = 0$ *for some* $x \in \mathbb{R}^d$*.*

Chiral field

The field $g\colon \mathbb{R}^d \to U(N)$ is a general chiral field. Its Dirihlet functional has a form

$$\frac{1}{2}\int_{\mathbb{R}^d} tr(\partial_\mu g^{-1}(x)\partial_\mu g(x))dx = -\frac{1}{2}\int_{\mathbb{R}^d} tr(Z_\mu(x)Z^\mu(x))dx, \quad (8)$$

where $Z_\mu = g^{-1}(x)\partial_\mu g(x)$. The equation of motion has a form

$$\sum_{\mu=1}^{d} \partial_\mu(g^{-1}(x)\partial_\mu g(x)) = 0$$

or

$$\begin{cases} divZ = 0 \\ \partial_\mu Z_\nu - \partial_\nu Z_\mu + [Z_\mu, Z_\nu] = 0, \end{cases}$$

where $Z = (Z_1, \ldots, Z_d)$

The Yang-Mills action functional can be represented as an infinite-dimensional analogue of the Dirichlet functional of chiral field. This analogue is also derived using Cèsaro averaging.

Theorem

Under some technical assumptions the following equality holds

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{\mu=1}^{d}\int_{\mathbb{R}^d} E(tr(\partial_{p_\mu h_k}U^x(b,1)^{-1}\partial_{p_\mu h_k}U^x(b,1))dx =$$

$$= -\int_{\mathbb{R}^d} tr(F_{\mu\nu}(x)F^{\mu\nu}(x))dx \quad (9)$$

Stochastic Lévy Laplacian and the derivative of the annihilation process

Consider the Gelfand triplet $\mathcal{E} \subset \Gamma(L_2([0,1],\mathbb{R}^d)) \subset \mathcal{E}^*$, where $\Gamma(L_2([0,1],\mathbb{R}^d))$ is the boson Fock space over the Hilbert space $L_2([0,1],\mathbb{R}^d)$),$\mathcal{E}$ is the space of white noise test functionals and $\mathcal{E}^*$ is the space of white noise generalized functionals. Due to the Wiener-Ito-Segal isomorphism the space $W_2^2(P)$ can be considered as a subspace of $\mathcal{E}^*$. On the $\mathcal{E}$ acts the family of so-called nonclassical Lévy Laplcians. The introduced stochastic Lévy Laplacian coincides with one element of this family (but not with the classical Lévy Laplacian). Moreover, the stochastic Lévy Laplacian can be represented as

$$\Delta_L = \lim_{\varepsilon \to 0} \int_{|s-t|<\varepsilon} \dot{a}_\mu(s)\dot{a}^\mu(t)dsdt,$$

where $\dot{a}^\mu(t)$ is the derivative of the annihilation process.

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