Ward and BPZ identities for Liouville quantum field theory on the Riemann sphere

Vincent Vargas [1]

ENS Paris

[1] based on joint works with: David, Kupiainen, Rhodes

Outline

1. Classical Liouville theory: A. Polyakov meets H. Poincaré
2. Building Liouville conformal field theory
3. Perspectives

Plan of the talk

1. Classical Liouville theory: A. Polyakov meets H. Poincaré
2. Building Liouville conformal field theory
3. Perspectives

A 1890 question of the Royal society of Science in Göttingen: metrics with negative curvature on the sphere?

Let $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ and $(z_i, \chi_i)_{1 \leq i \leq n} \in (\mathbb{C} \times \mathbb{R})^n$ with $\chi_i < 2$.

Existence and unicity of the Liouville equation with conical singularities (and $\phi_*(z) \underset{|z| \to \infty}{\sim} -4 \ln |z|$) :

$$\Delta_z \phi_*(z) = 2\pi \Lambda e^{\phi_*(z)}, \quad z \in \mathbb{C} \setminus \{z_1, \cdots, z_n\}$$

where $\Lambda > 0$ and

$$\phi_*(z) = \chi_i \ln \frac{1}{|z - z_i|} + O(1), \quad z \to z_i$$

First proof by: E. Picard in 1890!

Uniformisation theory: the legacy of Poincaré

H. Poincaré

Classical stress energy tensor

$$T_{cl}(z) = -\frac{1}{2}(\partial_z \phi_*(z))^2 + \partial^2_{zz}\phi_*(z)$$

Simple computation:

$$\partial_{\bar{z}} T_{cl} = 0, \quad z \in \mathbb{S}^2 \setminus \{z_1, \cdots, z_n\},$$

hence

$$T_{cl}(z) = \sum_{i=1}^{n} \frac{\chi_i(1-\chi_i/4)}{2(z-z_i)^2} + \sum_{i=1}^{n} \frac{a_i}{z-z_i}$$

where $(a_i)_{1\leq i\leq n}$ are the accessory parameters.

Uniformisation theory: the legacy of Poincaré

One has:

$$\pi \Lambda e^{\phi_*(z)} |dz|^2 = \frac{4|f'(z)|^2}{(1-|f(z)|^2)^2} |dz|^2 \quad \textbf{(Uniformisation theorem)}$$

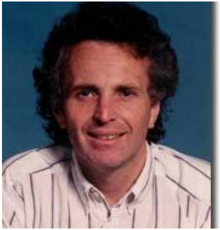
where $f = u_1/u_2$ and u_1, u_2 two independent solutions of the **Fuchsian** equation:

$$\frac{d^2 u}{dz^2}(z) + \frac{1}{2} T_{cl}(z) u(z) = 0 \qquad (u_1' u_2 - u_1 u_2' = 1)$$

In particular, $e^{-\phi_*/2}$ solves the **Fuchsian** PDE:

$$\partial_{zz}^2 e^{-\phi_*/2}(z) + \frac{1}{2} T_{cl}(z) e^{-\phi_*/2}(z) = 0 \quad \textbf{(classical degenerate field)}$$

The legacy of Polyakov



A. Polyakov

Quantum version of the Liouville equation:

A. Polyakov (1981): *Quantum geometry of bosonic strings*.

Main question for A. Polyakov: can one solve the quantum version? This lead to CFT...

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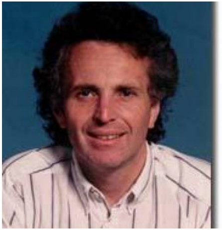

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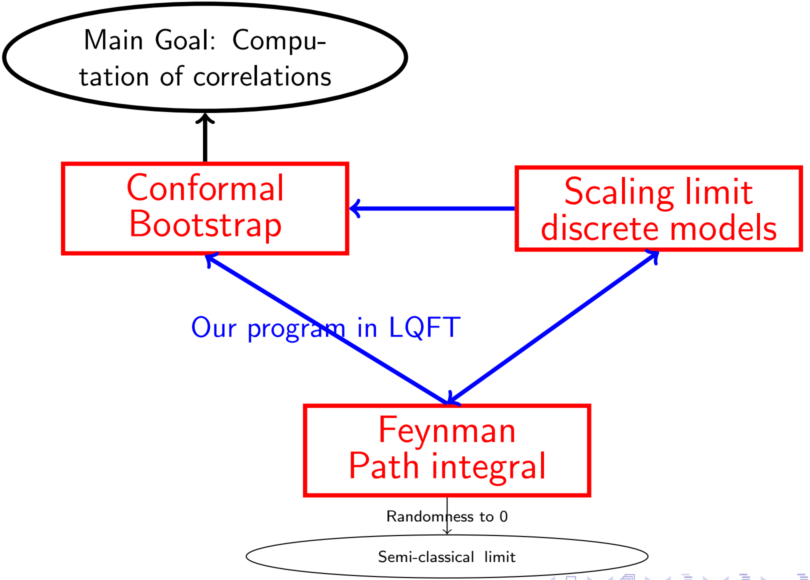
"Complex analysis in the quantum domain" (Polyakov, *From Quarks to Strings*):

A. Belavin, A. Polyakov, A. Zamolodchikov (1984): *Infinite conformal symmetry in two-dimensional quantum field theory*.

Our program: make sense out of these quantum objects!

Plan of the talk

What is building Liouville CFT?



LQFT on the Riemann sphere $\mathbb{S}^2$

Formal definition of correlations:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle := \int \left(\prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(X)} DX,$$

where

- DX "Lebesgue measure" on functional space
- S_L Liouville action:

  $$S_L(X) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(4|\partial_z X(z)|^2 + 2QX(z)g(z) + 4\pi\mu e^{\gamma X(z)} g(z) \right) dz$$

  with $g(z) = \frac{4}{(1+|z|^2)^2}$ round metric, $\gamma \in]0,2]$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.
- **Liouville field**: $\phi(z) = X(z) + \frac{Q}{2} \ln g(z)$.

Motivation from discrete planar maps

The KPZ relation reads:

$$\langle \prod_{i=4}^{n} \Phi(x_i) \rangle = \frac{\langle \prod_{i=4}^{n} \sigma(x_i) \rangle \langle e^{\gamma\phi(x_1)} e^{\gamma\phi(x_2)} e^{\gamma\phi(x_3)} \prod_{i=4}^{n} e^{\alpha\phi(x_i)} \rangle}{\langle e^{\gamma\phi(x_1)} e^{\gamma\phi(x_2)} e^{\gamma\phi(x_3)} \rangle}$$

where:

- $x_1, x_2, x_3 \in \mathbb{S}^2$ fixed points in the embedding of the map in the sphere
- σ scaling limit of primary field on **regular lattice** with conformal weight Δ_σ
- Φ scaling limit of same primary field on **planar map**
- Conformal weight condition: $\Delta_\sigma + \Delta_\alpha = 1$

Previous morning talks: $\sigma = 1$ with $\alpha = \gamma$ and $\gamma = \sqrt{\frac{8}{3}}$.

Existence of the correlation functions

Theorem (DKRV, 2014)

One can define the correlations $\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle$ *by a regularization procedure. The correlations are non trivial if and only if:*

$$\forall i,\ \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i > 2Q \qquad \text{(Seiberg bounds)}$$

In particular, existence implies $n \geq 3$*!*

Idea of proof: interpret the gradient term in Liouville action as Gaussian Free Field with average distributed as Lebesgue.

An explicit expression for the correlation functions

The existence is in fact based on the following explicit expression:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = A \prod_{j<k} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \mu^{-s} \Gamma(s) \mathbb{E}[Z_1^{-s}]$$

where $s = \frac{\sum_i \alpha_i - 2Q}{\gamma}$, A some constant (depending on the α_i and γ) and

$$Z_1 = \int_{\mathbb{C}} e^{\gamma X_g(z) - \frac{\gamma^2}{2}\mathbb{E}[X_g(z)^2]} \prod_l \frac{1}{|z - z_l|^{\gamma \alpha_l}} g(z)^{1 - \frac{\gamma}{4}\sum_l \alpha_l} dz$$

with X_g GFF with vanishing mean on the sphere.

The KPZ formula

Theorem (DKRV, 2014)

Let $(\alpha_i)_i$ satisfy the Seiberg bounds. If ψ is a Möbius transform, we have

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(\psi(z_i))} \rangle = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle$$

where $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ is the conformal weight of $e^{\alpha_i \phi(z)}$.

Reminder: $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

The first Ward identity

Quantum stress energy tensor: let $\phi_\epsilon = \phi * \theta_\epsilon$ where θ_ϵ smooth **isotropic** mollifiers tending to identity and set

$$T_\epsilon(z) = -(\partial_z \phi_\epsilon(z))^2 + c_\epsilon + Q\partial^2_{zz}\phi_\epsilon(z), \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}$$

where $c_\epsilon = \frac{c_1}{\epsilon^2} + c_2$.

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Theorem (KRV, 2015)

The following convergence defines the first Ward identity:

$$\langle T(z) \prod_{i=1}^n e^{\alpha_i \phi(z_i)} \rangle := \lim_{\epsilon \to 0} \langle T_\epsilon(z) \prod_{i=1}^n e^{\alpha_i \phi(z_i)} \rangle$$

$$= \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle \prod_{i=1}^n e^{\alpha_i \phi(z_i)} \rangle + \sum_k \frac{1}{(z - z_k)} \partial_{z_k} \langle \prod_{i=1}^n e^{\alpha_i \phi(z_i)} \rangle$$

The **quantum** Fuchsian equation: the BPZ differential equation of order 2

The field $e^{-\frac{\gamma}{2}\phi}$ satisfies the BPZ differential equation of order 2:

Theorem (KRV, 2015)

For all $(-\frac{\gamma}{2}, (\alpha_i)_i)$ *which satisfy the Seiberg bounds:*

$$\frac{4}{\gamma^2}\partial_{zz}^2\langle e^{-\frac{\gamma}{2}\phi(z)}\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle + \sum_{k=1}^n \frac{\Delta_{\alpha_k}}{(z-z_k)^2}\langle e^{-\frac{\gamma}{2}\phi(z)}\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle$$
$$+ \sum_{k=1}^n \frac{1}{z-z_k}\partial_{z_k}\langle e^{-\frac{\gamma}{2}\phi(z)}\prod_{i=1}^n e^{\alpha_i\phi(z_i)}\rangle = 0,$$

Remark: one also has the dual BPZ equation with the field $e^{-\frac{2}{\gamma}\phi(z)}$.

Applications of the BPZ equation of order 2

By conformal covariance:

$$\langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_\gamma(\alpha_1, \alpha_2, \alpha_3)$$

where $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc... and $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ structure constant.

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Corollaries of the BPZ equation in LQFT:

- Expressions for the 4 point functions $\langle e^{-\frac{\gamma}{2}\phi(z)} \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle$ in terms of hypergeometric functions (conformal blocks) and $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$.
- Non trivial functional relations on $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$: Teschner's trick!

Applications of the BPZ equation of order 2: Conformal blocks for the 4 point function

Reminder on hypergeometric ${}_2F_1$:

$$ {}_2F_1(a, b, c, z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n $$

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Corollary (KRV, 2015)

For all $(-\frac{\gamma}{2}, (\alpha_i)_{1\leq i\leq 3})$ which satisfy the Seiberg bounds

$$ \langle e^{-\frac{\gamma}{2}\phi(z)} e^{\alpha_1\phi(0)} e^{\alpha_1\phi(1)} e^{\alpha_1\phi(\infty)} \rangle = C_1 |F(z)|^2 + C_2 |\tilde{F}(z)|^2 $$

where $F, \tilde{F}$ are hypergeometric functions and C_1, C_2 constants which depend on $\gamma, \alpha_1, \alpha_2, \alpha_3$.

Applications of the BPZ equation of order 2: 3 point structure constant

Set $\bar{\alpha} = \sum_i \alpha_i$ and $l(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ with Γ standard Gamma function.

Corollary (KRV, 2015)

Let $(\alpha_1, \alpha_2, \alpha_3)$ be such that $\alpha_1 < Q - \frac{\gamma}{2}$ and $\sum_i \alpha_i > 2Q + \frac{\gamma}{2}$. Then we have the following relation

$$\frac{C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} = -\frac{1}{\pi\mu} \frac{l(-\frac{\gamma^2}{4}) l(\frac{\gamma\alpha_1}{2}) l(\frac{\alpha_1\gamma}{2} - \frac{\gamma^2}{4}) l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q)) l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2})) l(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2}))}.$$

Plan of the talk

Perspectives and open problems

Work in progress in the bootstrap program:

- OS positivity+ Second Ward identity+Construction of the Virasoro algebra of CFT.
- Higher order degenerate fields $e^{-n\frac{\gamma}{2}\phi}$ ($n \in \mathbb{N}^*$) should satisfy higher order BPZ equations.
- Prove the DOZZ proposal for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$.

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Other:

- Taking the semi-classical limit ($\alpha_i = \chi_i/\gamma$ and $\gamma \to 0$) and BPZ equation, give a probabilistic proof of the Polyakov conjecture for the accessory parameters (Lacoin, Rhodes, V.).
- Construct Liouville quantum gravity on higher genus surfaces (Guillarmou, Rhodes, V.).