Ward and BPZ identities for Liouville quantum field theory on the Riemann sphere

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Outline

1. Classical Liouville theory: A. Polyakov meets H. Poincaré

2. Building Liouville conformal field theory

3. Perspectives
Plan of the talk

1. Classical Liouville theory: A. Polyakov meets H. Poincaré

2. Building Liouville conformal field theory

3. Perspectives
A 1890 question of the Royal society of Science in Göttingen: metrics with negative curvature on the sphere?

Let \( S^2 = \mathbb{C} \cup \{ \infty \} \) and \((z_i, \chi_i)_{1 \leq i \leq n} \in (\mathbb{C} \times \mathbb{R})^n \) with \( \chi_i < 2 \).

Existence and unicity of the Liouville equation with conical singularities (and \( \phi^* (z) \sim -4 \ln |z| \) as \( |z| \to \infty \)) :

\[
\Delta_z \phi^* (z) = 2\pi \Lambda e^{\phi^* (z)}, \quad z \in \mathbb{C} \setminus \{ z_1, \cdots, z_n \}
\]

where \( \Lambda > 0 \) and

\[
\phi^* (z) = \chi_i \ln \frac{1}{|z - z_i|} + O(1), \quad z \to z_i
\]

First proof by: E. Picard in 1890!
Uniformisation theory: the legacy of Poincaré

**Classical stress energy tensor**

\[ T_{cl}(z) = -\frac{1}{2}(\partial_z \phi_*(z))^2 + \partial_{zz} \phi_*(z) \]

Simple computation:

\[ \partial_z T_{cl} = 0, \quad z \in S^2 \setminus \{z_1, \cdots, z_n\}, \]

hence

\[ T_{cl}(z) = \sum_{i=1}^{n} \frac{\chi_i(1 - \chi_i/4)}{2(z - z_i)^2} + \sum_{i=1}^{n} \frac{a_i}{z - z_i} \]

where \((a_i)_{1 \leq i \leq n}\) are the accessory parameters.
Uniformisation theory: the legacy of Poincaré

One has:

$$\pi \Lambda e^{\phi_*(z)}|dz|^2 = \frac{4|f'(z)|^2}{(1 - |f(z)|^2)^2}|dz|^2 \quad \text{(Uniformisation theorem)}$$

where $f = u_1/u_2$ and $u_1, u_2$ two independent solutions of the Fuchsian equation:

$$\frac{d^2 u}{dz^2}(z) + \frac{1}{2} T_{cl}(z) u(z) = 0 \quad (u'_1 u_2 - u_1 u'_2 = 1)$$

In particular, $e^{-\phi_*/2}$ solves the Fuchsian PDE:

$$\partial_{zz} e^{-\phi_*/2}(z) + \frac{1}{2} T_{cl}(z) e^{-\phi_*/2}(z) = 0 \quad \text{(classical degenerate field)}$$
Quantum version of the Liouville equation:

A. Polyakov (1981): *Quantum geometry of bosonic strings*.

Main question for A. Polyakov: can one solve the quantum version? This lead to CFT...
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“A. Polyakov

“Complex analysis in the quantum domain” (Polyakov, *From Quarks to Strings*):


Our program: make sense out of these quantum objects!
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What is building Liouville CFT?

Main Goal: Computation of correlations

Conformal Bootstrap

Scaling limit discrete models

Feynman Path integral

Randomness to 0

Semi-classical limit

Our program in LQFT
Formal definition of correlations:

\[
\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle := \int \left( \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(X)} DX,
\]

where

- \( DX \) "Lebesgue measure" on functional space
- \( S_L \) Liouville action:

\[
S_L(X) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left( 4|\partial_z X(z)|^2 + 2QX(z)g(z) + 4\pi \mu e^{\gamma X(z)}g(z) \right) dz
\]

with \( g(z) = \frac{4}{(1+|z|^2)^2} \) round metric, \( \gamma \in ]0, 2] \), \( Q = \frac{2}{\gamma} + \frac{\gamma}{2} \) and \( \mu > 0 \).

- Liouville field: \( \phi(z) = X(z) + \frac{Q}{2} \ln g(z) \).
The KPZ relation reads:

$$\langle \prod_{i=4}^{n} \Phi(x_i) \rangle = \frac{\langle \prod_{i=4}^{n} \sigma (x_i) \rangle \langle e^{\gamma \phi(x_1)} e^{\gamma \phi(x_2)} e^{\gamma \phi(x_3)} \prod_{i=4}^{n} e^{\alpha \phi(x_i)} \rangle}{\langle e^{\gamma \phi(x_1)} e^{\gamma \phi(x_2)} e^{\gamma \phi(x_3)} \rangle}$$

where:

- $x_1, x_2, x_3 \in S^2$ fixed points in the embedding of the map in the sphere
- $\sigma$ scaling limit of primary field on regular lattice with conformal weight $\Delta_\sigma$
- $\Phi$ scaling limit of same primary field on planar map
- Conformal weight condition: $\Delta_\sigma + \Delta_\alpha = 1$

Previous morning talks: $\sigma = 1$ with $\alpha = \gamma$ and $\gamma = \sqrt{\frac{8}{3}}$. 
Existence of the correlation functions

Theorem (DKRV, 2014)

One can define the correlations $\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle$ by a regularization procedure. The correlations are non trivial if and only if:

\[ \forall i, \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i > 2Q \quad (\text{Seiberg bounds}) \]

In particular, existence implies $n \geq 3!$

Idea of proof: interpret the gradient term in Liouville action as Gaussian Free Field with average distributed as Lebesgue.
An explicit expression for the correlation functions

The existence is in fact based on the following explicit expression:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = A \prod_{j<k} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \mu^{-s} \Gamma(s) \mathbb{E}[Z_1^{-s}]$$

where $s = \frac{\sum_i \alpha_i - 2Q}{\gamma}$, $\mu$ some constant (depending on the $\alpha_i$ and $\gamma$) and

$$Z_1 = \int_{\mathbb{C}} e^{\gamma X_g(z) - \frac{\gamma^2}{2} \mathbb{E}[X_g(z)^2]} \prod_l \frac{1}{|z - z_l|^{\gamma \alpha_l}} g(z)^{1-\frac{\gamma}{4} \sum_l \alpha_l} dz$$

with $X_g$ GFF with vanishing mean on the sphere.
The KPZ formula

**Theorem (DKRV, 2014)**

Let \((\alpha_i)_i\) satisfy the Seiberg bounds. If \(\psi\) is a Möbius transform, we have

\[
\langle \prod_{i=1}^{n} e^{\alpha_i \phi(\psi(z_i))} \rangle = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle
\]

where \(\Delta_{\alpha_i} = \frac{\alpha_i}{2} (Q - \frac{\alpha_i}{2})\) is the conformal weight of \(e^{\alpha_i \phi(z)}\).

Reminder: \(Q = \frac{\gamma}{2} + \frac{2}{\gamma}\).
The first Ward identity

Quantum stress energy tensor: let $\phi_\epsilon = \phi \ast \theta_\epsilon$ where $\theta_\epsilon$ smooth isotropic mollifiers tending to identity and set

$$T_\epsilon(z) = -\left( \partial_z \phi_\epsilon(z) \right)^2 + c_\epsilon + Q \partial^2_{zz} \phi_\epsilon(z), \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}$$

where $c_\epsilon = \frac{c_1}{\epsilon^2} + c_2$. 

Theorem (KRV, 2015) The following convergence defines the first Ward identity:
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Theorem (KRV, 2015)

The following convergence defines the first Ward identity:

\[
\langle T(z) \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle := \lim_{\epsilon \to 0} \langle T_\epsilon(z) \prod_{i=1}^{n} e^{\alpha_i \phi_\epsilon(z_i)} \rangle = \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle + \sum_k \frac{1}{(z - z_k)} \partial_{z_k} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle
\]
The **quantum** Fuchsian equation: the BPZ differential equation of order 2

The field $e^{-\frac{\gamma}{2} \phi}$ satisfies the BPZ differential equation of order 2:

**Theorem (KRV, 2015)**

For all $(-\frac{\gamma}{2}, (\alpha_i)_i)$ which satisfy the Seiberg bounds:

\[
\frac{4}{\gamma^2} \partial_{zz}^2 \langle e^{-\frac{\gamma}{2} \phi(z)} \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle + \sum_{k=1}^{n} \frac{\Delta \alpha_k}{(z - z_k)^2} \langle e^{-\frac{\gamma}{2} \phi(z)} \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle \\
+ \sum_{k=1}^{n} \frac{1}{z - z_k} \partial_{z_k} \langle e^{-\frac{\gamma}{2} \phi(z)} \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = 0,
\]

Remark: one also has the dual BPZ equation with the field $e^{-\frac{2}{\gamma} \phi(z)}$. 
By conformal covariance:

\[ \langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_{\gamma}(\alpha_1, \alpha_2, \alpha_3) \]

where \( \Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2} \) etc... and \( C_{\gamma}(\alpha_1, \alpha_2, \alpha_3) \) structure constant.
Applications of the BPZ equation of order 2

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Corollaries of the BPZ equation in LQFT:

- Expressions for the 4 point functions \( \langle e^{-\frac{\gamma}{2} \phi(z)} \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle \) in terms of hypergeometric functions (conformal blocks) and \( C_\gamma(\alpha_1, \alpha_2, \alpha_3) \).

- Non trivial functional relations on \( C_\gamma(\alpha_1, \alpha_2, \alpha_3) \) : Teschner's trick!
Applications of the BPZ equation of order 2: Conformal blocks for the 4 point function

Reminder on hypergeometric $\,_{2}F_{1}$:

$$\,_{2}F_{1}(a, b, c, z) = \sum_{n \geq 0} \frac{(a)_n(b)_n}{(c)_nn!} z^n$$
Applications of the BPZ equation of order 2: Conformal blocks for the 4 point function

Reminder on hypergeometric $\mathbf{2F1}$:

$$\mathbf{2F1}(a, b, c, z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

Corollary (KRV, 2015)

For all $(-\frac{\gamma}{2}, (\alpha_i)_{1 \leq i \leq 3})$ which satisfy the Seiberg bounds

$$\langle e^{-\frac{\gamma}{2} \phi(z)} e^{\alpha_1 \phi(0)} e^{\alpha_1 \phi(1)} e^{\alpha_1 \phi(\infty)} \rangle = C_1 |F(z)|^2 + C_2 |\tilde{F}(z)|^2$$

where $F, \tilde{F}$ are hypergeometric functions and $C_1, C_2$ constants which depend on $\gamma, \alpha_1, \alpha_2, \alpha_3$. 
Set $\bar{\alpha} = \sum_i \alpha_i$ and $l(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ with $\Gamma$ standard Gamma function.

**Corollary (KRV, 2015)**

Let $(\alpha_1, \alpha_2, \alpha_3)$ be such that $\alpha_1 < Q - \frac{\gamma}{2}$ and $\sum_i \alpha_i > 2Q + \frac{\gamma}{2}$. Then we have the following relation

$$\frac{C_\gamma(\alpha_1 + \frac{\gamma}{2}, \alpha_2, \alpha_3)}{C_\gamma(\alpha_1 - \frac{\gamma}{2}, \alpha_2, \alpha_3)} = -\frac{1}{\pi \mu} \frac{l\left(-\frac{\gamma^2}{4}\right)l\left(\frac{\gamma\alpha_1}{2}\right)l\left(\frac{\alpha_1 \gamma}{2} - \frac{\gamma^2}{4}\right)l\left(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2})\right)}{l\left(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q)\right)l\left(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2})\right)l\left(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2})\right)}.$$
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Work in progress in the bootstrap program:

- OS positivity + Second Ward identity + Construction of the Virasoro algebra of CFT.
- Higher order degenerate fields $e^{-n\frac{\gamma}{2}\phi}$ ($n \in \mathbb{N}^*$) should satisfy higher order BPZ equations.
- Prove the DOZZ proposal for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$. Other:
  - Taking the semi-classical limit ($\alpha_i = \chi_i / \gamma$ and $\gamma \to 0$) and BPZ equation, give a probabilistic proof of the Polyakov conjecture for the accessory parameters (Lacoin, Rhodes, V.).
  - Construct Liouville quantum gravity on higher genus surfaces (Guillarmou, Rhodes, V.).
Perspectives and open problems

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