On connection between two sets of higher-dimensional gamma matrices and a primitive field equation

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Dirac gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}\mathbf{1}, \quad a, b = 0, 1, 2, 3.$$

$$\eta = \|\eta^{ab}\| = \mathrm{diag}(1, -1, -1, -1).$$

Dirac P.A.M., Proc. Roy. Soc. Lond. A117 (1928).

Dirac P.A.M., Proc. Roy. Soc. Lond. A118 (1928).

Pauli's fundamental theorem

Theorem (Pauli)

Consider 2 sets of square complex matrices

$$\gamma^a, \qquad \beta^a, \qquad a = 1, 2, 3, 4.$$

of size 4*. Let these 2 sets satisfy the following conditions*

$$\begin{aligned} \gamma^a\gamma^b + \gamma^b\gamma^a &= 2\eta^{ab}\mathbf{1}, \qquad \eta = \mathrm{diag}(1, -1, -1, -1), \\ \beta^a\beta^b + \beta^b\beta^a &= 2\eta^{ab}\mathbf{1}. \end{aligned}$$

Then there exists a unique (up to multiplication by a complex constant) complex matrix T such that

$$\gamma^a = T^{-1}\beta^a T, \qquad a = 1, 2, 3, 4.$$

W.Pauli, *Contributions mathematiques a la theorie des matrices de Dirac*, Ann. Inst. Henri Poincare 6, (1936).

Clifford algebras $\mathcal{C}\ell^{\mathbb{R}}(p,q)$ and $\mathcal{C}\ell^{\mathbb{C}}(p,q) = \mathbb{C}\otimes\mathcal{C}\ell^{\mathbb{R}}(p,q)$

linear space E over $\mathbb{R}$, $\dim E = 2^n$, basis: $\{e, e^{a_1}, e^{a_1 a_2}, \ldots, e^{1\ldots n}\}$,

$1 \leq a_1 < \cdots < a_k \leq n$, multiplication:

1. distributivity, associativity, e - identity element,
2. $e^{a_1} \ldots e^{a_k} = e^{a_1 \ldots a_k}$, $1 \leq a_1 < \cdots < a_k \leq n$,
3. $e^a e^b + e^b e^a = 2\eta^{ab} e$, $\eta = ||\eta^{ab}|| = \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$, $p+q=n$.

$$\mathcal{C}\ell(p,q) \ni U = ue + \sum_a u_a e^a + \sum_{a<b} u_{ab} e^{ab} + \cdots + u_{1\ldots n} e^{1\ldots n} = u_A e^A.$$

$$\mathcal{C}\ell(p,q) = \bigoplus_{k=0}^{n} \mathcal{C}\ell_k(p,q), \qquad \mathcal{C}\ell_k(p,q) = \{\sum_{|A|=k} u_A e^A\}.$$

$$\mathcal{C}\ell(p,q) = \mathcal{C}\ell_{\mathrm{Even}}(p,q) \oplus \mathcal{C}\ell_{\mathrm{Odd}}(p,q),$$

$$\mathcal{C}\ell_{\mathrm{Even}}(p,q) = \bigoplus_{k-even} \mathcal{C}\ell_k(p,q), \qquad \mathcal{C}\ell_{\mathrm{Odd}}(p,q) = \bigoplus_{k-odd} \mathcal{C}\ell_k(p,q).$$

Theorem

$$cen\mathcal{C}\!\ell(p,q)=\begin{cases}\mathcal{C}\!\ell_0(p,q), & \textit{if } n \textit{ - even;}\\ \mathcal{C}\!\ell_0(p,q)\oplus\mathcal{C}\!\ell_n(p,q), & \textit{if } n \textit{ - odd.}\end{cases}$$

Theorem (Cartan 1908, Bott 1960)

$$\mathcal{C}\!\ell^{\mathbb{R}}(p,q)\simeq\begin{cases}\mathrm{Mat}(2^{\frac{n}{2}},\mathbb{R}), & \textit{if } p-q\equiv 0;2 \mod 8;\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{R})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{R}), & \textit{if } p-q\equiv 1 \mod 8;\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \textit{if } p-q\equiv 3;7 \mod 8;\\ \mathrm{Mat}(2^{\frac{n-2}{2}},\mathbb{H}), & \textit{if } p-q\equiv 4;6 \mod 8;\\ \mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H})\oplus\mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H}), & \textit{if } p-q\equiv 5 \mod 8.\end{cases}$$

Theorem

$$\mathcal{C}\!\ell^{\mathbb{C}}(p,q)\simeq\begin{cases}\mathrm{Mat}(2^{\frac{n}{2}},\mathbb{C}), & \textit{if } n \textit{ - even;}\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \textit{if } n \textit{ - odd.}\end{cases}$$

Let the set of Clifford algebra elements satisfies the conditions

$$\beta^a \in \mathcal{C}\!\ell(p,q), \qquad \beta^a\beta^b + \beta^b\beta^a = 2\eta^{ab}e.$$

Then the set

$$\gamma^a = T^{-1}\beta^a T, \qquad \forall \text{ invertible } T \in \mathcal{C}\!\ell(p,q)$$

satisfies the conditions

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}e.$$

Really,

$$\gamma^a\gamma^b + \gamma^b\gamma^a = T^{-1}\beta^a TT^{-1}\beta^b T + T^{-1}\beta^b TT^{-1}\beta^a T =$$
$$= T^{-1}(\beta^a\beta^b + \beta^b\beta^a)T = T^{-1}2\eta^{ab}eT = 2\eta^{ab}e.$$

Theorem (Case of even n)

Consider real (or complexified) Clifford algebra $C\ell(p,q)$ of even dimension $n=p+q$. Let the following 2 sets of Clifford algebra elements γ^a, β^a, $a=1,2,\ldots,n$ satisfy conditions

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}e,\quad \beta^a\beta^b+\beta^b\beta^a=2\eta^{ab}e.$$

Then both sets of elements generate bases of Clifford algebra and there exists a unique (up to multiplication by a real (complex) constant) element $T\in C\ell(p,q)$ such that

$$\gamma^a=T^{-1}\beta^a T,\qquad \forall a=1,\ldots,n.$$

Moreover, we can always find this element T in the form

$$T=\sum_A \beta^A F\gamma_A,\qquad \gamma_A=(\gamma^A)^{-1}$$

where F is any element of a set

$$1)\,\{\gamma^A, A\in\mathcal{I}_{\mathrm{Even}}\}\quad \textit{if}\,\beta^{1\ldots n}\neq-\gamma^{1\ldots n};\qquad 2)\,\{\gamma^A, A\in\mathcal{I}_{\mathrm{Odd}}\}\quad \textit{if}\,\beta^{1\ldots n}\neq\gamma^{1\ldots n}$$

such that corresponding T is nonzero $T\neq0$.

Case of even n in matrix formalism

Theorem

Let n - even and 2 sets of square matrices $\gamma^a, \beta^a, a = 1, 2, \ldots, n$ satisfy conditions

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbf{1}, \qquad \beta^a\beta^b + \beta^b\beta^a = 2\eta^{ab}\mathbf{1}.$$

- *If matrices are complex of the order $2^{\frac{n}{2}}$, then there exists a unique (up to a complex constant) matrix T such that*
- *If signature is $p - q \equiv 0, 2 \mod 8$ and matrices are real of the order $2^{\frac{n}{2}}$, then there exists a unique (up to a real constant) matrix T such that*
- *If signature is $p - q \equiv 4, 6 \mod 8$ and matrices are over the quaternions of the order $2^{\frac{n-2}{2}}$, then there exists a unique (up to a real constant) matrix T such that*

$$\gamma^a = T^{-1}\beta^a T, \qquad a = 1, \ldots n.$$

The case of odd n

Example 1: $\mathcal{C}\!\ell^{\mathbb{R}}(2,1) \simeq \mathrm{Mat}(2,\mathbb{R}) \oplus \mathrm{Mat}(2,\mathbb{R})$ with generators e^1, e^2, e^3. We can take

$$\gamma^1 = e^1, \qquad \gamma^2 = e^2, \qquad \gamma^3 = e^1 e^2.$$

Then $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbf{1}$. Elements $\gamma^1, \gamma^2, \gamma^3$ generate not $\mathcal{C}\!\ell^{\mathbb{R}}(2,1)$, but generate $\mathcal{C}\!\ell^{\mathbb{R}}(2,0) \simeq \mathrm{Mat}(2,\mathbb{R})$.
Example 2: $\mathcal{C}\!\ell^{\mathbb{R}}(3,0) \simeq \mathrm{Mat}(2,\mathbb{C})$ with generators e^1, e^2, e^3.

$$\beta^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta^2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta^3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\gamma^a = -\sigma^a, \qquad a = 1,2,3.$$

Then $\gamma^{123} = -\beta^{123}$. Suppose, that we have such $T \in \mathrm{GL}(2,\mathbb{C})$ that $\gamma^a = T^{-1}\beta^a T$. Then

$$\gamma^{123} = T^{-1}\beta^1 T T^{-1}\beta^2 T T^{-1}\beta^3 T = T^{-1}\beta^1\beta^3\beta^3 T = \beta^{123}$$

and we obtain a contradiction (we use that $\beta^{123} = \sigma^{123} = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = i\mathbf{1}$).

But we have such element $T = \mathbf{1}$ that $\gamma^a = -T^{-1}\beta^a T$.

Theorem

Consider real (or complexified) Clifford algebra $\mathcal{C}\ell(p,q)$ *of odd dimension* $n=p+q$. *Let the following 2 sets of Clifford algebra elements* γ^a, β^a, $a=1,2,\ldots,n$ *satisfy conditions*

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}e,\quad \beta^a\beta^b+\beta^b\beta^a=2\eta^{ab}e.$$

Then in the case of Clifford algebra $\mathcal{C}\ell(p,q)$ *of signature* $p-q\equiv 1 \mod 4$ *elements* $\gamma^{1\ldots n}$ *and* $\beta^{1\ldots n}$ *equals* $\pm e^{1\ldots n}$ *and then corresponding sets generate bases of Clifford algebra or equals* $\pm e$ *and then corresponding sets don't generate bases.*

In the case of Clifford algebra $\mathcal{C}\ell(p,q)$ *of signature* $p-q\equiv 3 \mod 4$ *elements* $\gamma^{1\ldots n}$ *and* $\beta^{1\ldots n}$ *equals* $\pm e^{1\ldots n}$, *and then corresponding sets generate bases of Clifford algebra or (only for* $\mathcal{C}\ell^{\mathbb{C}}(p,q)$*) equals* $\pm ie$ *and then corresponding sets don't generate bases.*

There exists a unique (up to a invertible element of Clifford algebra center) element T such that

$$
\begin{array}{llll}
1) & \gamma^a = T^{-1}\beta^a T, & \forall a=1,\ldots,n & \Leftrightarrow \quad \beta^{1\ldots n} = \gamma^{1\ldots n},\\
2) & \gamma^a = -T^{-1}\beta^a T, & \forall a=1,\ldots,n & \Leftrightarrow \quad \beta^{1\ldots n} = -\gamma^{1\ldots n},\\
3) & \gamma^a = e^{1\ldots n}T^{-1}\beta^a T, & \forall a=1,\ldots,n & \Leftrightarrow \quad \beta^{1\ldots n} = e^{1\ldots n}\gamma^{1\ldots n},\\
4) & \gamma^a = -e^{1\ldots n}T^{-1}\beta^a T, & \forall a=1,\ldots,n & \Leftrightarrow \quad \beta^{1\ldots n} = -e^{1\ldots n}\gamma^{1\ldots n},\\
5) & \gamma^a = ie^{1\ldots n}T^{-1}\beta^a T, & \forall a=1,\ldots,n & \Leftrightarrow \quad \beta^{1\ldots n} = ie^{1\ldots n}\gamma^{1\ldots n},\\
6) & \gamma^a = -ie^{1\ldots n}T^{-1}\beta^a T, & \forall a=1,\ldots,n & \Leftrightarrow \quad \beta^{1\ldots n} = -ie^{1\ldots n}\gamma^{1\ldots n}.
\end{array}
$$

Note, that all 6 cases can be written in the form $\gamma^a = (\beta^{1\ldots n}\gamma_{1\ldots n})T^{-1}\beta^a T$.

Moreover, in the case of real Clifford algebra $\mathcal{C}\!\ell^{\mathbb{R}}(p,q)$ of signature $p-q\equiv 3 \mod 4$ we can always find this element T in the form

$$\sum_{A\in\mathcal{I}_{\mathrm{Even}}} \beta^A F \gamma_A,$$

where F is any element of the set

$$\gamma^A, \qquad A\in\mathcal{I}_{\mathrm{Even}},$$

such that corresponding T is nonzero $T\neq 0$.
In the case of real Clifford algebra $\mathcal{C}\!\ell^{\mathbb{R}}(p,q)$ of signature $p-q\equiv 1 \mod 4$ and complexified Clifford algebra $\mathcal{C}\!\ell^{\mathbb{C}}(p,q)$ we can always find this element T in the form

$$\sum_{A\in\mathcal{I}_{\mathrm{Even}}} \beta^A F \gamma_A,$$

where F is one of the elements of the set

$$\gamma^A+\gamma^B, \qquad A, B\in\mathcal{I}_{\mathrm{Even}}.$$

Clifford field vectors

$$\mathbb{R}^{p,q}, \quad p+q=n, \quad \eta = \|\eta_{\mu\nu}\|, \quad \mu,\nu = 1,\ldots,n$$
$$x^\mu \to \acute{x}^\mu = p^\mu_\nu x^\nu, \quad O(p,q) = \{P = \|p^\mu_\nu\| \in \mathrm{Mat}(n,\mathbb{R}) : P^T \eta P = \eta\}.$$

Tensor fields with values in Clifford algebra: $U^{\mu_1\cdots\mu_r}_{\nu_1\ldots\nu_s} \in \mathcal{C}\!\ell(p,q)\mathrm{T}^r_s$ where components $U^{\mu_1\cdots\mu_r}_{\nu_1\ldots\nu_s}$ are considered as functions $\mathbb{R}^{p,q} \to \mathcal{C}\!\ell(p,q)$.
If $h^\mu = h^\mu(x)$ are components of vector field with values in $\mathcal{C}\!\ell(p,q)$

$$h^\mu(x) = u^\mu(x)e + u^\mu_a(x)e^a + \sum_{a_1<a_2} u^\mu_{a_1 a_2}(x)e^{a_1 a_2} + \ldots + u^\mu_{1\ldots n}(x)e^{1\ldots n},$$

that satisfy the following relations:

$$h^\mu(x)h^\nu(x) + h^\nu(x)h^\mu(x) = 2\eta^{\mu\nu}e, \quad \mu,\nu = 1,\ldots,n \tag{1}$$

for any $\forall x \in \mathbb{R}^{p,q}$ and the condition

$$\pi_0(h^1 \ldots h^n) = 0, \tag{2}$$

then the vector $h^\mu \in \mathcal{C}\!\ell(p,q)\mathrm{T}^1$ is called a *Clifford field vector*.
We call an algebra with the basis $\{e, h^\mu, h^{\mu\nu}, \ldots, h^{1\ldots n}\}$ an algebra of h-forms $\mathcal{C}\!\ell[h](p,q)$. It is a generalization of the Atiyah-Kähler algebra.

Theorem (Local generalized Pauli's theorem in the case of even n)

Let n be even number and $h^a = h^a(x)$, $a = 1, \ldots, n$ are functions $\Omega \to \mathcal{C}\ell(p,q)$ of class $C^k(\Omega)$ such that

$$h^a(x)h^b(x) + h^b(x)h^a(x) = 2\eta^{ab}e, \quad a, b = 1, \ldots, n, \quad \forall x \in \Omega.$$

Then for any $x_0 \in \Omega$ there exists $\varepsilon > 0$ and there exists a function $T = T(x) : O_\varepsilon(x_0) \to \mathcal{C}\ell(p,q)$, satisfying the conditions

1. *$T(x)$ – function of class $C^k(O_\varepsilon(x_0))$;*
2. *$T(x)$ – an invertible element of Clifford algebra $\mathcal{C}\ell(p,q)$ for any $x \in O_\varepsilon(x_0)$;*
3. *$e^a = T^{-1}(x)h^a(x)T(x)$, $a = 1, \ldots, n$, $\forall x \in O_\varepsilon(x_0)$;*
4. *The function $T(x)$ is defined up to multiplication by (real in the case $\mathcal{C}\ell^{\mathbb{R}}(p,q)$ or complex in the case $\mathcal{C}\ell^{\mathbb{C}}(p,q)$) function of class $C^k(O_\varepsilon(x_0))$ that is not equal to zero for any point of $O_\varepsilon(x_0)$.*

Primitive field equation, gauge invariance

Lie algebra: $\mathcal{C}\ell_{\circledS}(p,q)=\mathcal{C}\ell(p,q)\setminus \mathrm{cen}\mathcal{C}\ell(p,q)$.

Theorem

Let $h^\nu\in\mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}^1$ be a Clifford field vector and $C_\mu\in\mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$ satisfy the primitive field equation

$$\partial_\mu h_\rho-[C_\mu,h_\rho]=0,\quad \forall\mu,\rho=1,\ldots,n.$$

Let $S:\mathbb{R}^{p,q}\to\mathcal{C}\ell^\times(p,q)$ be a function with values in $\mathcal{C}\ell^\times(p,q)$ such that

$$S^{-1}\partial_\mu S\in\mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1.$$

Then, the following components of covectors

$$\acute{h}_\rho=S^{-1}h_\rho S\in\mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1,\quad \acute{C}_\mu=S^{-1}C_\mu S-S^{-1}\partial_\mu S\in\mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$$

also satisfy the equation

$$\partial_\mu\acute{h}_\rho-[\acute{C}_\mu,\acute{h}_\rho]=0,\quad \forall\mu,\rho=1,\ldots,n.$$

General solution of primitive field equation

Theorem

Let $h^\nu \in \mathcal{C}\!\ell_{\circledS}(p,q)\mathrm{T}^1$ be a Clifford field vector and $C_\mu \in \mathcal{C}\!\ell_{\circledS}(p,q)\mathrm{T}_1$. Then the following two systems of equations are equivalent:

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0, \quad \mu,\rho = 1,\ldots,n \quad \Leftrightarrow \quad C_\mu = \sum_{k=1}^{\acute{n}} \mu_k \pi[h]_k((\partial_\mu h^\rho) h_\rho), \tag{3}$$

where $\acute{n} = n$ for even n, $\acute{n} = n-1$ for odd n and

$$\mu_k = \frac{1}{n-(-1)^k(n-2k)}.$$

In the case $n = 2$

$$C_\mu = \sum_{k=1}^{2} \mu_k \pi[h]_k((\partial_\mu h^\rho)h_\rho) = \frac{1}{2}\pi[h]_1((\partial_\mu h^\rho)h_\rho) + \frac{1}{4}\pi[h]_2((\partial_\mu h^\rho)h_\rho)$$

$$= \frac{1}{2}(\partial_\mu h^\rho)h_\rho - \frac{1}{16}h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha - \frac{3}{32}h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta.$$

In the case $n = 3$

$$C_\mu = \sum_{k=1}^{2} \mu_k \pi[h]_k((\partial_\mu h^\rho)h_\rho) = \frac{1}{4}\pi[h]_1((\partial_\mu h^\rho)h_\rho) + \frac{1}{4}\pi[h]_2((\partial_\mu h^\rho)h_\rho)$$

$$= \frac{1}{4}\pi[h]_{12}((\partial_\mu h^\rho)h_\rho) = \frac{3}{16}(\partial_\mu h^\rho)h_\rho - \frac{1}{16}h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha.$$

In the case $n = 4$

$$\begin{aligned} C_\mu &= \sum_{k=1}^{4} \mu_k \pi[h]_k((\partial_\mu h^\rho)h_\rho) = \frac{1}{6}\pi[h]_1((\partial_\mu h^\rho)h_\rho) + \frac{1}{4}\pi[h]_2((\partial_\mu h^\rho)h_\rho) \\ &\quad + \frac{1}{2}\pi[h]_3((\partial_\mu h^\rho)h_\rho) + \frac{1}{8}\pi[h]_4((\partial_\mu h^\rho)h_\rho) \\ &= \frac{1}{4}(\partial_\mu h^\rho)h_\rho + \frac{67}{576}h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha + \frac{73}{2304}h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta \\ &\quad - \frac{19}{2304}h^\gamma h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta h_\gamma - \frac{25}{9216}h^\delta h^\gamma h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta h_\gamma h_\delta. \end{aligned}$$

Global Pauli's theorem and primitive field equation

Example: $n=2$, $(p,q)=(2,0)$, $\gamma^a=e^a$, $\beta^a=h^a(x)$, $a=1,2$.
From $\partial_\mu e_\rho-[C_\mu,e_\rho]=0$ we obtain $C_\mu=0$. From $\partial_\mu h_\rho-[\acute{C}_\mu,h_\rho]=0$ in particular case $h^\mu=u^\mu_a(x)e^a\in\mathcal{C}\ell_1(p,q)$ we obtain $\acute{C}_\mu=\frac{1}{4}(\partial_\mu h^\rho)h_\rho$.
Using gauge invariance

$$\acute{h}_\rho=S^{-1}e_\rho S,\quad \acute{C}_\mu=S^{-1}C_\mu S-S^{-1}\partial_\mu S$$

we obtain equation $\acute{C}_\mu=-S^{-1}\partial_\mu S$ or $\partial_\mu(S^{-1})=\acute{C}_\mu S^{-1}$.
If

$$h^1(x)=\cos\varphi(x)e^1+\sin\varphi(x)e^2,\qquad h^2(x)=-\sin\varphi(x)e^1+\cos\varphi(x)e^2,$$

then

$$\acute{C}_\mu=\frac{1}{4}(\partial_\mu h^\rho)h_\rho=-\frac{\partial_\mu\varphi}{2}e^{12}.$$

We obtain $\partial_\mu(S^{-1})=-\frac{\partial_\mu\varphi}{2}e^{12}S^{-1}$ and $S^{-1}=\exp(\frac{-\varphi}{2}e^{12})C,\quad C\in\mathcal{C}\ell(p,q)$.
Finally,

$$S(x)=\exp(\frac{\varphi}{2}e^{12})=\cos\frac{\varphi}{2}e+\sin\frac{\varphi}{2}e^{12},\qquad h^\rho(x)=S^{-1}(x)e^\rho S(x).$$

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Thank you for your attention!