A conformal bootstrap approach to the Potts model

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based on work with Marco Picco and Raoul Santachiara, `arXiv:1607`

Abstract: We study four-point functions of the Potts model, with two-dimensional critical percolation as a special case. We propose an exact ansatz for the spectrum: an infinite, discrete and non-diagonal combination of representations of the Virasoro algebra. Based on this ansatz, we compute four-point functions using a numerical conformal bootstrap approach. The results agree with Monte-Carlo computations of connectivities of random clusters.

Contents

1 Connectivities of random clusters

In the random cluster formulation of Fortuin and Kasteleyn, the Potts model is a theory of graphs on a square lattice. The connected components of a graph are called clusters, and the probability of a graph $\mathcal{G}$ is defined as

$$\text{Probability}(\mathcal{G}) = q^{\#\,\text{clusters}} p^{\#\,\text{bonds}} (1-p)^{\#\,\text{edges without bond}} \quad , \quad (q \in \mathbb{C}) \ . \tag{1}$$

The model becomes conformally invariant when the bond probability p takes the critical value $p_c = \frac{\sqrt{q}}{\sqrt{q}+1}$, and the size of the lattice becomes infinite. It can be described by a conformal fied theory with the central charge

$$c = 1 - 6\left(\beta - \frac{1}{\beta}\right)^2 \quad , \quad q = 4\cos^2 \pi\beta^2 \ , \tag{2}$$

with in particular $q = 1, c = 0$ for critical percolation.

The observables that we want to measure are probabilities that a number of points belong to the same cluster. The simplest example is the two-point connectivity

$$P(z_1, z_2) \propto |z_1 - z_2|^{-4\Delta_{(0,\frac{1}{2})}} , \tag{3}$$

where the critical exponent $\Delta_{(0,\frac{1}{2})}$ will be given later. Here I want to focus on the four-point connectivities

$$P_0(\{z_i\}) = \text{Probability}(z_1, z_2, z_3, z_4 \text{ are all in the same cluster}) , \tag{4}$$

$$P_1(\{z_i\}) = \text{Probability}(z_1, z_2 \text{ and } z_3, z_4 \text{ are in two different clusters}) , \tag{5}$$

$$P_2(\{z_i\}) = \text{Probability}(z_1, z_3 \text{ and } z_2, z_4 \text{ are in two different clusters}) , \tag{6}$$

$$P_3(\{z_i\}) = \text{Probability}(z_1, z_4 \text{ and } z_2, z_3 \text{ are in two different clusters}) , \tag{7}$$

Permutations of $\{z_i\}$ leave P_0 invariant, and map P_1, P_2, P_3 to one another.

This is not just for the sake of doing something that was not done before. In contrast to two- and three-point connectivities, four-point connectivities have a non-trivial dependence on a geometric parameter, namely the cross-ratio

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} . \tag{8}$$

From this dependence, it is in principle possible to extract the spectrum and operator product expansion of the underlying CFT, and to check its consistency. It is fair to say that

Understanding a CFT ⇔ understanding its four-point functions.

2 CFT interpretation

We want to relate the connectivities P_σ to CFT four-point functions. The behaviour of P_σ under permutations of $\{z_i\}$ suggest that we look for four-point functions of the type

$$R_0 = \left< V_+ V_+ V_+ V_+ \right> , \tag{9}$$

$$R_1 = \left< V_+ V_+ V_- V_- \right> , \tag{10}$$

$$R_2 = \left< V_+ V_- V_+ V_- \right> , \tag{11}$$

$$R_3 = \left< V_+ V_- V_- V_+ \right> , \tag{12}$$

where both $V_\pm$ are diagonal primary fields with the (left and right) conformal dimension $\Delta_{(0,\frac{1}{2})}$. We assume that there is a $\mathbb{Z}_2$ symmetry such that V_+ is even, and V_- is odd, so that for instance $\left< V_- V_- V_- V_+ \right> = 0$.

In order to compute the four-point functions, we need to know which primary fields appear in the OPEs

$$V_+V_+ = V_-V_- = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_0} V_{\Delta,\bar{\Delta}} \ , \tag{13}$$

$$V_+V_- = V_-V_+ = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_1} V_{\Delta,\bar{\Delta}} \ . \tag{14}$$

Then we have

$$R_0 = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_0} C_{\Delta,\bar{\Delta}} \mathcal{F}^{(s)}_{\Delta}(z)\mathcal{F}^{(s)}_{\bar{\Delta}}(\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_0} C_{\Delta,\bar{\Delta}} \mathcal{F}^{(t)}_{\Delta}(z)\mathcal{F}^{(t)}_{\bar{\Delta}}(\bar{z}) \ , \tag{15}$$

$$R_2 = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_1} D_{\Delta,\bar{\Delta}} \mathcal{F}^{(s)}_{\Delta}(z)\mathcal{F}^{(s)}_{\bar{\Delta}}(\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_1} D_{\Delta,\bar{\Delta}} \mathcal{F}^{(t)}_{\Delta}(z)\mathcal{F}^{(t)}_{\bar{\Delta}}(\bar{z}) \ , \tag{16}$$

for some coefficients $C_{\Delta,\bar{\Delta}}$ and $D_{\Delta,\bar{\Delta}}$ that are determined by the equality of the s- and t-channel decompositions. This equality, called crossing symmetry, is actually an overdetermined equation for the coefficients, and the existence of a nonzero solution is a strong constraint on the spectrums $\mathcal{S}_0$ and $\mathcal{S}_1$.

3 Ansatz for the spectrum

So we need to find a good ansatz for the spectrum, starting with the ground state – the state with the lowest total dimension $\Delta + \bar{\Delta}$, which dominates the OPE when two fields come close. The four-point connectivity P_0 manifestly tends to a three-point connectivity when two points come together, which suggests

1. Ground state $(\Delta, \bar{\Delta}) = (\Delta_{(0,\frac{1}{2})}, \Delta_{(0,\frac{1}{2})})$.

Moreover, single-valuedness of correlation functions implies that all states have half-integer spins,

$$s = \Delta - \bar{\Delta} \in \mathbb{Z} \ . \tag{17}$$

This is in particular satisfied by states with $\Delta = \bar{\Delta}$, called spinless or diagonal states. However, the spectrum $\mathcal{S}_1$ cannot be purely diagonal. Actually, if only even spins appeared in $\mathcal{S}_1$, then R_1, R_2, R_3 would be symmetric under permutations. (This is a nontrivial feature of conformal blocks.) So we assume

2. Presence of odd spins.

The dimension $\Delta_{(0,\frac{1}{2})}$ is a special case of

$$\Delta_{(r,s)} = \frac{c-1}{24} + \frac{1}{4}\left(r\beta - \frac{s}{\beta}\right)^2 \ . \tag{18}$$

These dimensions play a special role in 2d CFT, as they correspond to the so-called degenerate representations of the Virasoro algebra if r, s are positive integers. This leads us to assume

3. $\Delta, \bar{\Delta} \in \{\Delta_{(r,s)}\}_{r\in\mathbb{Z}, s\in\frac{1}{2}\mathbb{Z}}$.

How do we build non-diagonal spectrums from such representations? Using the identity $\Delta_{(r,-s)} = \Delta_{(r,s)} + rs$, it is tempting to use states of the type $(\Delta_{(r,s)}, \Delta_{(r,-s)})$ with $rs \in \frac{1}{2}\mathbb{Z}$. So we look for spectrums of the type

$$\mathcal{S}_{X,Y} = \left\{(\Delta_{(r,s)}, \Delta_{(r,-s)})\right\}_{r\in X, s\in Y} \ . \tag{19}$$

A spectrum that fits all our requirements is

$$\mathcal{S}_1 = \mathcal{S}_{2\mathbb{Z}, \mathbb{Z}+\frac{1}{2}} \ . \tag{20}$$

4 Numerics

First of all we should check that our proposed spectrum satisfies crossing symmetry. We write the crossing symmetry equation as

$$\sum_{(\Delta,\bar{\Delta})\in\mathcal{S}_1} D_{\Delta,\bar{\Delta}} \left(\mathcal{F}^{(s)}_{\Delta}(z)\mathcal{F}^{(s)}_{\bar{\Delta}}(\bar{z}) - \mathcal{F}^{(t)}_{\Delta}(z)\mathcal{F}^{(t)}_{\bar{\Delta}}(\bar{z})\right) = 0 \ . \tag{21}$$

These infinitely many equations (parametrized by z) with infinitely many unknowns $D_{\Delta,\bar{\Delta}}$ can be truncated to a finite system by truncating the spectrum to the N states with the lowest total dimensions, and taking $N-1$ positions $z_1, \cdots, z_{N-1}$. (We normalize the ground state structure constant to one.) The spectrum is consistent if the resulting structure constants $D_{\Delta,\bar{\Delta}}(N)$ are independent from the choice of $z_1, \cdots, z_{N-1}$ when $N \to \infty$, i.e. if the limit $D_{\Delta,\bar{\Delta}} = \lim_{N\to\infty} D_{\Delta,\bar{\Delta}}(N)$ exists.

We find (**notebook**) that our spectrum is consistent. The code is available on GitHub, and can be used for testing other ansatzes – for example, $\mathcal{S}_{2\mathbb{Z}+1,\mathbb{Z}}$ is crossing-symmetric too! The code enables us to compute the four-point functions R_1, R_2, R_3 with a very good precision, $O(10^{-9})$ maybe.

Then we should compare the results with Monte-Carlo simulations. The simulations are done on a lattice of size 8192, for $1 \leq q \leq 3$, and the relative error is $O(10^{-3})$. For any value of q that we investigated, we found (σ, z-independent) numbers λ, μ such that

$$R_\sigma(z) = \lambda\left(P_0(z) + \mu P_\sigma(z)\right) \qquad , \qquad (\sigma = 1, 2, 3) \ . \tag{22}$$

So the conformal bootstrap analysis successfully computes three of the four four-point connectivities. What is missing is the fourth combination R_0.

5 Outlook

For which values of q and c are our results valid? On the CFT side, the only constraint is the convergence of the decomposition of four-point functions into conformal blocks. This requires that the real part of the total dimensions is bounded from below in the spectrum $\mathcal{S}_1$. The total conformal is

$$\Delta_{(r,s)} + \Delta_{(r,-s)} = \frac{c-1}{12} + \frac{1}{2}\left(r^2\beta^2 + \frac{s^2}{\beta^2}\right) . \tag{23}$$

So we need

$$\Re\beta^2 > 0 \quad \Leftrightarrow \quad \Re c < 13 . \tag{24}$$

This covers all complex values of q. However, on the statistical physics side, complex values of q have not been studied. Actually, our Monte-Carlo simulations are limited to $1 \leq q \leq 3$ i.e. $0 \leq c \leq \frac{4}{5}$.

In the case $q = 4, c = 1$, the four-state Potts model is a special case of the Ashkin–Teller model, that was solved by Alexey Zamolodchikov. And he proved that the spectrum $\mathcal{S}_1 = \mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$ is crossing symmetric. Moreover, he found

$$\mathcal{S}_0 \underset{c=1}{=} \mathcal{S}_{2\mathbb{Z},\mathbb{Z}} . \tag{25}$$

This answer makes sense for any value of c, but we find that it works only at $c = 1$.

In the case $q = 3, c = \frac{4}{5}$, the spectrum $\mathcal{S}_1 = \mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$ reduces to the non-diagonal sector of the non-diagonal minimal model with $\beta^2 = \frac{5}{6}$. This sector is finite, and it involves the six fields

$V_{(0,\frac{1}{2})}$	$V_{(3,2)}$	σ	$\frac{1}{15}$
$V_{(0,\frac{3}{2})}$	$V_{(3,1)}$	Z	$\frac{2}{3}$
$V_{(2,\frac{1}{2})}$	$V_{(1,2)}$	ϵ	$\frac{2}{5}$
$V_{(-2,\frac{1}{2})}$	$V_{(1,3)}$	X	$\frac{7}{5}$
$V_{(2,\frac{3}{2})}$	$V_{(1,1)}$	1	0
$V_{(-2,\frac{3}{2})}$	$V_{(1,4)}$	Y	3

(26)

This suggests that the missing spectrum $\mathcal{S}_0$ is given by the diagonal sector of this model. But this looks very different from the situation at $c = 1$.

(Notice that in the case $c = \frac{1}{2}$, our four-point connectivities do not agree with minimal model four-point functions. This is because the minimal model OPE for our field $V_{(0,\frac{1}{2})} = V_{(2,1)}$ yields the fields $V_{(1,1)}$ and $V_{(3,1)}$. Our ground state $V_{(2,1)}$, and more generally the states in our spectrum $\mathcal{S}_1 = \mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$, are absent.)

Finally, it would be good to solve the relevant CFTs analytically rather than numerically, using the same methods that were used for solving other CFTs such as Liouville theory. Some steps in this direction have been taken by Estienne and Ikhlef. This is work in progress [Santiago Migliaccio + S. R.].