Comparing oscillator and prefundamental representations

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Introduction

Quantum group approach

- A quantum group of special type — quantum loop algebra $\mathcal{G}$.
- The universal R-matrix $\mathcal{R} \in \mathcal{G} \otimes \mathcal{G}$.

Integrability objects

- Two representations φ, ψ of $\mathcal{G}$ on the spaces V and U for two factors.
- The representation space of the first factor representation — auxiliary space.
- The representation space of the second factor representation — quantum space.
- Monodromy operator

$$M = (\varphi \otimes \psi)(\mathcal{R}).$$

- Transfer operator

$$T = \operatorname{tr}_V M.$$

- Similar story about L-operators and Q-operators.

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Lie algebras $\mathfrak{sl}_{l+1}$ and beyond

- We start with the finite dimensional complex simple Lie algebras $\mathfrak{sl}_{l+1}$.
- The Cartan subalgebra

$$\mathfrak{h} = \bigoplus_{i=1}^{l} \mathbb{C} H_i, \qquad H_i = E_{ii} - E_{i+1,i+1}.$$

- The loop algebra of $\mathfrak{sl}_{l+1}$

$$\mathcal{L}(\mathfrak{sl}_{l+1}) = \mathbb{C}[\zeta, \zeta^{-1}] \otimes \mathfrak{sl}_{l+1}.$$

- The standard central extension

$$\widetilde{\mathcal{L}}(\mathfrak{sl}_{l+1}) = \mathcal{L}(\mathfrak{sl}_{l+1}) \oplus \mathbb{C}\, c.$$

- The central extension of the Cartan subalgebra

$$\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\, c = \mathbb{C}\, h_0 \oplus \bigoplus_{i=1}^{l} \mathbb{C}\, h_i = \bigoplus_{i=0}^{l} \mathbb{C}\, h_i,$$

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Simple roots

- The simple roots $\alpha_i \in \widetilde{\mathfrak{h}}^*$ are defined by the equality

$$\langle \alpha_i, h_j \rangle = a_{ji}, \qquad i, j = 0, 1, \ldots, l,$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

is the extended Cartan matrix of $\mathfrak{sl}_{l+1}$.

Quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

- The deformation parameter $q \in \mathbb{C}^\times$ is not a root of unity.
- The Drinfeld–Jimbo generators e_i, f_i, $i = 0, 1, \dots l$, and q^x, $x \in \widetilde{\mathfrak{h}}$.
- The relations

$$q^0 = 1, \qquad q^{x_1} q^{x_2} = q^{x_1 + x_2}, \qquad q^{\nu c} = 1, \quad \nu \in \mathbb{C},$$
$$q^x e_i \, q^{-x} = q^{\langle \alpha_i, x \rangle} e_i, \qquad q^x f_i \, q^{-x} = q^{-\langle \alpha_i, x \rangle} f_i,$$
$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.$$

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Second Drinfeld's realization

- The new generators
  - ⋆ $\xi^{\pm}_{i,n}$, where $i = 1, \dots l$ and $n \in \mathbb{Z}$;
  - ⋆ $\chi_{i,n}$, where $i = 1, \dots l$ and $n \in \mathbb{Z}^{\times}$;
  - ⋆ q^x, $x \in \widetilde{\mathfrak{h}}$.
- For the isomorphism see (Khoroshkin, Tolstoy 1992; Beck 1994) etc. In particular,

$$\xi^{+}_{i,0} = e_i, \qquad \xi^{-}_{i,0} = f_i.$$

- An infinite dimensional commutative subalgebra is generated by all $\chi_{i,n}$ and q^x.
- Another set of commuting generators is formed by $\phi^{+}_{i,n}$, $i = 1, \dots, l$, $n \geq 0$ and $\phi^{-}_{i,n}$, $i = 1, \dots, l$, $n \leq 0$, determined by the equation

$$\sum_{n=0}^{\infty} \phi^{\pm}_{i,\pm n} u^{\pm n} = q_i^{\pm h_i} \exp\left(\pm (q - q^{-1}) \sum_{n=1}^{\infty} \chi_{i,\pm n} u^{\pm n} \right)$$

One also assumes that $\phi^{+}_{i,n} = 0$ for $n < 0$ and $\phi^{-}_{i,n} = 0$ for $n > 0$.

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Representations of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

Jimbo's homomorphism

- The Jimbo's homomorphism $\epsilon\colon \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1})) \to \mathrm{U}_q(\mathfrak{gl}_{l+1})$.

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- The Drinfeld–Jimbo generators $E_i, F_i, i = 1, \ldots l+1$, and q^X, $X \in \mathfrak{k}$.
- The relations

$$q^0 = 1, \qquad q^{X_1} q^{X_2} = q^{X_1+X_2},$$
$$q^X E_i q^{-X} = q^{\langle \alpha_i, X\rangle} E_i, \qquad q^X F_i q^{-X} = q^{-\langle \alpha_i, X\rangle} F_i,$$
$$[E_i, F_j] = \delta_{ij} \frac{q^{K_i - K_{i+1}} - q^{-K_i + K_{i+1}}}{q - q^{-1}}.$$

- The Serre relations (explicit form is not important here).

Jimbo's homomorphism

- The Jimbo's homomorphism $\epsilon\colon \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1})) \to \mathrm{U}_q(\mathfrak{gl}_{l+1})$.

Quantum group $\mathrm{U}_q(\mathfrak{gl}_{l+1})$

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Representations of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

Highest weight representation π^λ of $U_q(\mathfrak{gl}_{l+1})$

- A weight representation

$$V^\lambda = \bigoplus_{\mu \in \mathfrak{k}^*} V_\mu,$$

where

$$V_\mu = \{v \in V^\lambda \mid q^X v = q^{\langle \mu, X \rangle} v\}.$$

- The highest weight vector $v^\lambda \in V^\lambda$:

$$E_i v^\lambda = 0, \quad i = 1, \ldots, l, \qquad q^X v^\lambda = q^{\langle \lambda, X \rangle} v^\lambda, \quad i = 1, \ldots, l+1.$$

- The highest weight vector generates the representation space:

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Highest weight representation φ^λ of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$

- The composition $\varphi = \pi^\lambda \circ \epsilon$ endows V^λ with the structure of a $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$-module.
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$$\epsilon(e_i) = E_i, \qquad \epsilon(q^{h_i}) = q^{K_i - K_{i+1}}, \qquad i = 1, \ldots, l,$$

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Refining the weight decomposition

- The infinite dimensional commutative subalgebra generated by $\phi^{\pm}_{i,n}$.
- The generalized eigenvectors

$$(\phi^{\pm}_{i,n} - \Psi^{\pm}_{i,n})^p v = 0.$$

- An ℓ-weight

$$\boldsymbol{\Psi} = \{\Psi^{+}_{i,n} \in \mathbb{C} \mid i = 1, \ldots l, n \in \mathbb{Z}_+\} \cup \{\Psi^{-}_{i,-n} \in \mathbb{C} \mid i = 1, \ldots l, n \in \mathbb{Z}_+\}.$$

- The generating functions

$$\Psi^{+}_{i}(u) = \sum_{n \in \mathbb{Z}_+} \Psi^{+}_{i,n} u^n, \qquad \Psi^{-}_{i}(u^{-1}) = \sum_{n \in \mathbb{Z}_+} \Psi^{-}_{i,-n} u^{-n}.$$

- A gradation automorphism

$$\Gamma_\zeta(e_i) = \zeta^{s_i} e_i, \qquad \Gamma_\zeta(f_i) = \zeta^{-s_i} f_i, \qquad \Gamma_\zeta(q^x) = q^x.$$

- Twisting a representation φ:

$$\varphi \to \varphi_\zeta = \varphi \circ \Gamma_\zeta, \qquad \Psi^{+}(u) \to \Psi^{+}(\zeta^s u), \qquad \Psi^{-}(u^{-1}) \to \Psi^{-}(\zeta^{-s} u^{-1}),$$

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Quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_2))$

Representation $(\varphi^{(\lambda_1, \lambda_2)})_\zeta$

- The highest ℓ-weights

$$\Psi^+(u) = q^{\lambda_1 - \lambda_2} \frac{1 - q^{2\lambda_2}\zeta^s u}{1 - q^{2\lambda_1}\zeta^s u}, \qquad \Psi^-(u) = q^{-\lambda_1 + \lambda_2} \frac{1 - q^{-2\lambda_2}\zeta^{-s} u^{-1}}{1 - q^{-2\lambda_1}\zeta^{-s} u^{-1}}.$$

- The Kirillov–Reshetikhin modules $\lambda = (k, 0)$

$$\Psi^+(u) = q^{k} \frac{1 - \zeta^s u}{1 - q^{2k}\zeta^s u}, \qquad \Psi^-(u) = q^{-k} \frac{1 - \zeta^{-s} u^{-1}}{1 - q^{-2k}\zeta^{-s} u^{-1}}.$$

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Quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_3))$

Representation $(\varphi^{(\lambda_1, \lambda_2, \lambda_3)})_\zeta$

- The highest ℓ-weights

$\Psi_1^+(u)$	$\Psi_2^+(u)$
$q^{\lambda_1-\lambda_2} \dfrac{1-q^{2\lambda_2}u}{1-q^{2\lambda_1}u}$	$q^{\lambda_2-\lambda_3} \dfrac{1-q^{2\lambda_3-1}u}{1-q^{2\lambda_2-1}u}$

$\Psi_1^-(u^{-1})$	$\Psi_2^-(u^{-1})$
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Representations of the Borel subalgebra $U_q(\mathfrak{b}^+)$

- The Borel subalgebra $U_q(\mathfrak{b}^+)$ of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ is generated by e_i, $i = 0, 1, \ldots, l$, and q^x, $x \in \widetilde{\mathfrak{h}}$.
- There is no a simple way to define $U_q(\mathfrak{b}^+)$ in terms of the new Drinfeld generators. However, one has

$$\phi^+_{i,n} \in U_q(\mathfrak{b}^+), \qquad i = 1, \ldots, l.$$

  and we define ℓ-weights via $\Psi^+_i(u)$.
- Given $\xi \in \widetilde{\mathfrak{h}}^*$, a shifted representation

$$\varphi[\xi](e_i) = \varphi(e_i), \quad i = 0, 1, \ldots l, \qquad \varphi[\xi](q^x) = q^{\langle \xi, x \rangle} \varphi(q^x), \quad x \in \widetilde{\mathfrak{h}}.$$

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Prefundamental and oscillator representations

Prefundamental representations

- Definition

$$\Psi_i^+(u) = (\underbrace{1, \ldots, 1}_{i-1}, (1 - au)^{\pm 1}, \underbrace{1, \ldots, 1}_{l-i}), \qquad i = 1, \ldots, l, \quad a \in \mathbb{C}^\times.$$

- Can be obtained as inductive limits of certain inductive systems constructed from Kirillov–Reshetikhin modules (Hernandez, Jimbo 2012).
- For $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ there are $2l$ representations.

Oscillator representaions

- For the case of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ can be obtained as a 'limit' of V^λ when $\lambda_i - \lambda_{i+1} \to -\infty$ and then applying the automorphisms related to the symmetries of the Dynkin diagram (Bazhanov, Hibbed, Khoroshkin 2001; Nirov, Razumov 2016).
- For $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ we have two representation, and for $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$, $l > 1$, we have $2l + 2$ representations.

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Oscillator representations

- The representation $(\rho_1)_\zeta$

$$\Psi^+(u) = 1 - \zeta^s q u$$

- The representation $(\rho_2)_\zeta$

$$\Psi^+(u) = (1 - \zeta^s q^3 u)^{-1}$$

Quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_3))$

Oscillator representations

	$\Psi_1^+(u)$	$\Psi_2^+(u)$
$(\rho_1)_\zeta$	$q^{-3}(1-\zeta^s q^{-2}u)^{-1}$	1
$(\rho_2)_\zeta$	$q(1-\zeta^s u)$	$q^{-2}(1-\zeta^s q^{-1}u)^{-1}$
$(\rho_3)_\zeta$	1	$1-\zeta^s qu$
$(\bar{\rho}_1)_\zeta$	$1+\zeta^s qu$	1
$(\bar{\rho}_2)_\zeta$	$q^{-2}(1+\zeta^s q^{-1}u)^{-1}$	$q(1+\zeta^s u)$
$(\bar{\rho}_3)_\zeta$	1	$q^{-3}(1+\zeta^s q^{-2}u)^{-1}$

Conclusion

Benefits

- The oscillator representations are not less prefundamental than genuine prefundamental ones.
- The simple explicit form of the representations.

Deficiency

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Oscillator representations

Algebra Osc_q

- The generators b, $b^\dagger$ and $q^{\nu N}$, $\nu \in \mathbb{C}$.
- The relations

$$q^0 = 1, \qquad q^{\nu_1 N} q^{\nu_2 N} = q^{(\nu_1+\nu_2)N},$$
$$q^{\nu N} b^\dagger q^{-\nu N} = q^{\nu} b^\dagger, \qquad q^{\nu N} b q^{-\nu N} = q^{-\nu} b,$$
$$b^\dagger b = [N]_q, \qquad b b^\dagger = [N+1]_q.$$

- The representation χ^+

$$q^{\nu N} v_m = q^{\nu m} v_m,$$
$$b^\dagger v_m = v_{m+1}, \qquad b\, v_m = [m]_q v_{m-1}.$$

- The representation χ^-

$$q^{\nu N} v_m = q^{-\nu(m+1)} v_m,$$
$$b\, v_m = v_{m+1}, \qquad b^\dagger\, v_m = -[m]_q v_{m-1}.$$

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Oscillator representations

General case

- For the case of arbitrary l we use $\mathrm{Osc}^{\otimes l}$ and define

$$b_i = 1 \otimes \ldots \otimes b \otimes \ldots \otimes 1, \qquad b_i^\dagger = 1 \otimes \ldots \otimes b^\dagger \otimes \ldots \otimes 1,$$
$$q^{\nu N_i} = 1 \otimes \ldots \otimes q^{\nu N} \otimes \ldots \otimes 1.$$

- The "Jimbo's homomorphism"

$$\rho(q^{\nu h_0}) = q^{\nu(2N_1 + \sum_{j=2}^{l} N_j)}, \quad \rho(q^{\nu h_i}) = q^{\nu(N_{i+1} - N_i)}, \quad \rho(q^{\nu h_l}) = q^{-\nu(2N_l + \sum_{j=1}^{l-1} N_j)},$$
$$\rho(e_0) = b_1^\dagger q^{\sum_{j=2}^{l} N_j}, \quad \rho(e_i) = -b_i b_{i+1}^\dagger q^{N_i - N_{i+1} - 1}, \quad \rho(e_l) = -\kappa_q^{-1} b_l q^{N_l},$$

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On French – Russia (USSR) collaboration

On July 7, 1967, the French Prime Minister Georges Pompidou visited the Institute of High Energy Physics, Protvino, and signed a cooperation agreement beetween IHEP and CEA center at Saclay.

On French – Russia (USSR) collaboration

Then a liquid-hydrogen bubble chamber "Mirabel" was designed and build in Saclay, and moved to Protvino. More then 50 French specialists with families came to Protvino to conduct researches.