Advances in Knot Polynomials

21 October 2016

ABSTRACT

Review of achievements and problems in the theory of colored knot polynomials. Accent is on the current mystery around the differential expansion and Racah matrices (6j-symbols) in rectangular representations.

J.W.Alexander, Trans.Amer.Math.Soc. **30** (2) (1928) 275-306

V.F.R.Jones, Invent.Math. **72** (1983) 1

P.Freyd, D.Yetter, J.Hoste, W.B.R.Lickorish, K.Millet, A.Ocneanu, Bull. AMS. **12** (1985) 239

J.H.Przytycki and K.P.Traczyk, Kobe J. Math. **4** (1987) 115-139

L.Kauffman,Topology **26** (1987) 395

Colored HOMFLY polynomial:

$$\boxed{\mathcal{H}_R^{\mathcal{K}}(A,q) = \left\langle \mathrm{Tr}_R \; P \exp \oint_{\mathcal{K}} \mathcal{A} \right\rangle}$$

average with CS action with the gauge group $G = SL(N)$:

$$\kappa \int_M \mathrm{Tr} \left(\mathcal{A} d \mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right)$$

$$q = \exp \frac{2\pi i}{\kappa + N}$$

$$A = q^N$$

HOMFLY

are

exactly-calculable

non-perturbative averages

in gauge QFT

For simply-connected M (R^3 or S^3):

- Calculation of HOMFLY
- Properties of HOMFLY
- Generalizations of HOMFLY
- Relations to other theories

Other M

- HOMFLY for virtual knots

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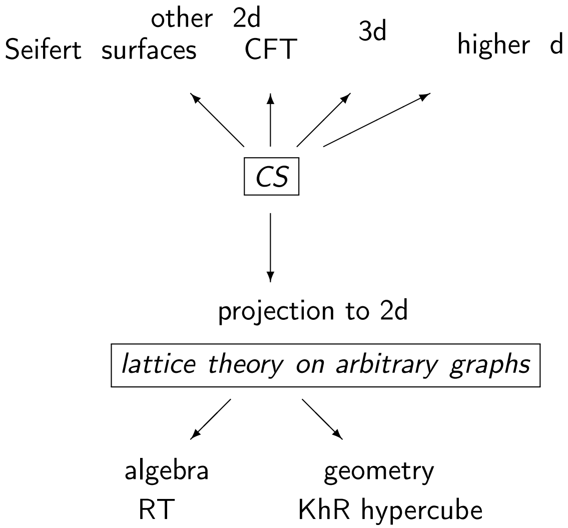
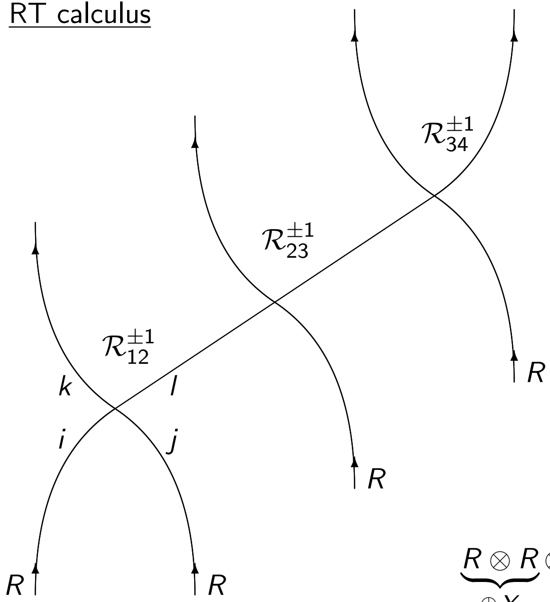
Calculation of HOMFLY

- Modernized Reshetikhin-Turaev calculus
- Paths in representation graphs
- Eigenvalue hypothesis
- Arborescent knots – new effective theory
- Fingered braids – the most efficient tool at the moment
- Evolution/family method
- ...


other 2d
Seifert surfaces
CFT
3d
higher d
CS
projection to 2d
lattice theory on arbitrary graphs
algebra
RT
geometry
KhR hypercube

RT calculus



traditional

$$\mathrm{tr}\ q^{\rho}\ \ldots$$

modern

$$\sum_Q D_Q \cdot \mathrm{Tr}_{W_Q}\ \ldots$$

$$\mathcal{R}_{23}^{Q} = U_{23}^{Q} \mathcal{R}_{12}^{Q} U_{23}^{Q\,\dagger}$$

$$\mathcal{R}_{12}^{Q} = \mathrm{diag}\Big(\epsilon_Y \cdot q^{\varkappa_Y}\Big)$$

$$\underbrace{R \otimes R}_{\oplus Y} \otimes R \otimes R = \oplus\ Q \otimes\ W_Q$$

Fundamental representation $R = \square$:

- skein relations $\qquad \mathcal{R} - \mathcal{R}^{-1} = q - q^{-1}$
- paths in representation tree

Other representations R:

- cabling method
- eigenvalue hypothesis
- tree calculus – under construction
- not just braids

BRAID CALCULUS

Needed is entire collection of mixing matrices

For 3 strands needed are only Racah matrices $\mathcal{S}^{(Q)}_{Y'Y''}$

$$\Big((\underbrace{R\otimes R}_{Y'})\otimes R \longrightarrow Q\Big) \;\longrightarrow\; \Big(R\otimes(\underbrace{R\otimes R}_{Y''}) \longrightarrow Q\Big)$$

but "inclusive": for all $Q \in R^{\otimes 3}$

This is realistic, but too few knots are 3-strand

ARBORESCENT KNOTS/LINKS

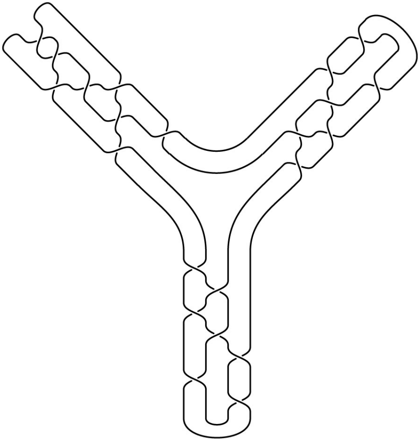
made from

fingers, propagators and vertices

ARBORESCENT (double-fat) KNOTS/LINKS

made from

fingers, propagators and vertices



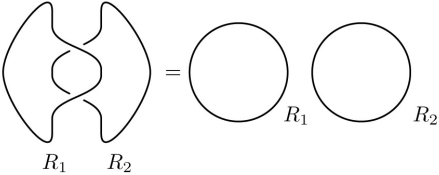
Needed are just two "exclusive" Racah matrices

$$\bar{S}: \quad \Big((R\otimes\bar{R})\otimes R \longrightarrow R\Big) \ \longrightarrow\ \Big(R\otimes(\bar{R}\otimes R)\longrightarrow R\Big)$$

and

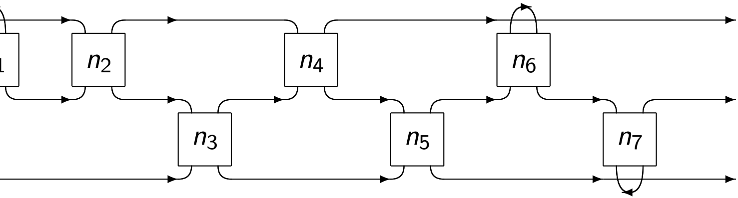
$$S: \quad \Big((\bar{R}\otimes R)\otimes R \longrightarrow R\Big) \ \longrightarrow\ \Big(\bar{R}\otimes(R\otimes R)\longrightarrow R\Big)$$

$$\bar{T}\bar{S}\bar{T} = ST^{-1}S^{\dagger}$$



T and $\bar{T}$ are diagonal matrices

FINGERED BRAIDS



FINGERED BRAIDS

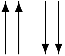
allow to handle more complicated knots
by using less strands

Three-strand fingered braids are already quite rich

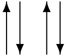
but even for them calculus is still hard *side stories:*

- The structure of the space of knots
- Effective gauge field theory for arborescent knots
- Gauge invariance, vertices and loops
- Rectangular and non-rectangular representations

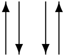
States:



σ

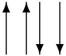


φ

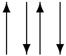


ϕ

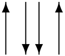
and conjugates:



σ^*



φ^*



ϕ^*

Each of them carries indices $\sigma_{AB} \longrightarrow \sigma_{X_{\alpha\beta}}$ with the gauge group acting by two orthogonal matrices $\mathcal{A}$ and $\mathcal{B}$:

$$\sigma_{X,\alpha,\beta} \longrightarrow \sum_{\alpha'\beta'} \mathcal{A}_{\alpha\alpha'} \mathcal{B}_{\beta\beta'} \sigma_{X,\alpha'\beta'}$$

Quadratic terms in the Lagrangian are:

- "local"ones

$$\sigma_X T_X^n \sigma_X = \sigma_{X,\alpha\beta} T^n_{X,\alpha\alpha'} \sigma_{X,\alpha'\beta} \qquad \varphi_X \bar{T}_X^{2n} \varphi_X,$$

$$\phi_X \bar{T}_X^{2n} \phi_X, \qquad \varphi_X \bar{T}_X^{2n-1} \phi_X, \qquad \phi_X \bar{T}_X^{2n} \varphi_X$$

plus conjugates,

- "non-local"ones

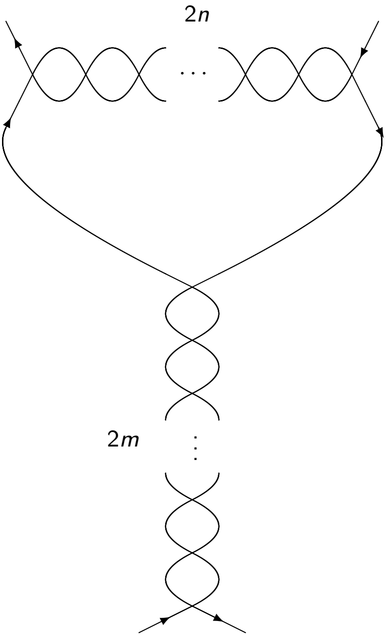
$$\sigma_X^* S_{XY}^\dagger \phi_Y, \qquad \phi_X^* S_{XY} \sigma_Y, \qquad \varphi_X^* \bar{S}_{XY} \varphi_Y$$

(note that there are no terms $\phi_X^* \phi_Y$)

Topologically allowed vertices are

$$\Gamma^{(1)} \sim \sigma_X^3, \qquad \Gamma^{(2)} \sim \varphi_X^3, \qquad \Gamma^{(3)} \sim \phi_X^2 \varphi_X$$

The problem is, however, to deal with the Greek indices in $\Gamma_{\alpha,\beta,\gamma}\Phi_{\alpha,\beta}\Phi_{\beta,\gamma}\Phi_{\gamma,\alpha}$. A naive anzatz like $\mathrm{tr}\ \sigma_X^3$ with the trace in Greek indices would be good for a transformation law $\sigma \longrightarrow \mathcal{A}\sigma\mathcal{A}^\dagger$, but it violates $\sigma \longrightarrow \mathcal{A}\sigma\mathcal{B}$ with independent $\mathcal{A}$ and $\mathcal{B}$. This means that at the representational level one can not get a gauge invariant description of our knot polynomials. If one calculates the Feynman diagram for some particular choice of S (in a particular gauge), the answer differs in other gauges so that there should be some "handy"compensational rule attached to the answer.


2n
. . .
2m
⋮

EVOLUTION for twist and double braid knots

$$H_R^{(m,n)} = \sum_{\mu,\nu \in R\otimes\bar{R}} \frac{\sqrt{D_\mu D_\nu}}{D_R}\ \bar{S}_{\mu\nu}\ \Lambda_\mu^{2m}\Lambda_\nu^{2n}$$

Properties of HOMFLY

- Polynomiality and integralities
- Factorizations
- Equations
- Hurwitz integrability
- Vogel's universality (unification of E_8-sectors of all Lie algebras)
- Differential expansions
- ...

$\dots$

DIFFERENTIAL EXPANSION

$$H^{4_1}_{[1]} = A^2 - q^2 + 1 - q^{-2} + A^{-2}$$

$$\boxed{= 1 + \{Aq\}\{A/q\}}$$

$$A = q^N$$

$$\{x\} = x - 1/x$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\{q^n\}}{\{q\}}$$

Differentials $\{Aq^n\} \sim [N + n]$

$$H^{4_1}_{[1]} = 1 + \{Aq\}\{A/q\}$$

$$H^{4_1}_{[2]} = 1 + [2]\{Aq^2\}\{A/q\} + \{Aq^3\}\{Aq^2\}\{A\}\{A/q\}$$

$$H^{4_1}_{[3]} = 1 + [3]\{Aq^3\}\{A/q\} + [3]\{Aq^4\}\{Aq^3\}\{A\}\{A/q\} + \\ + \{Aq^5\}\{Aq^4\}\{Aq^3\}\{Aq\}\{A\}\{A/q\}$$

$$\ldots$$

Differential expansion

Equations ↖ … ↑ ↗ Superpolynomials

$$H^{4_1}_{[1]} = 1 + \{Aq\}\{A/q\}$$

$$H^{4_1}_{[2]} = 1 + [2]\{Aq^2\}\{A/q\} + \{Aq^3\}\{Aq^2\}\{A\}\{A/q\}$$

↙ ↘

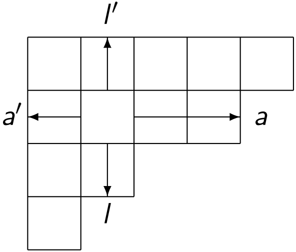
Other representations

$$H^{4_1}_{[r^s]} = \sum_{\lambda\in[r^s]} D_{\tilde\lambda}(r) D_\lambda(s) Z^\lambda_{r|s}$$

Other knots

$$H^{\mathcal{K}}_{[1]} = 1 + G^{\mathcal{K}}_{[1]}(q,A)\{Aq\}\{A/q\}$$

$$Z^{\lambda}_{r|s}(A,q) = \prod_{\Box\in\lambda} \{Aq^{r+a'(\Box)-l'(\Box)}\}\{Aq^{-s+a'(\Box)-l'(\Box)}\}$$



$$D_\lambda(N) = \prod_{\Box\in\lambda} \frac{\{Aq^{-l'(\Box)+a'(\Box)}\}}{\{q^{a(\Box)+l(\Box)+1}\}} = \prod_{\Box\in\lambda} \frac{[N-l'(\Box)+a'(\Box)]}{[a(\Box)+l(\Box)+1]}$$

Other knots:

$$H_{[1]}^{\mathcal{K}} = 1 + G_{[1]}^{\mathcal{K}}(q,A)\{Aq\}\{A/q\}$$

$$H_{[2]}^{\mathcal{K}} = 1 + [2]G_{[1]}^{\mathcal{K}}(q,A)\{Aq^2\}\{A/q\} + G_{[2]}^{\mathcal{K}}(q,A)\{Aq^2\}\{A\}\{A/q\}$$

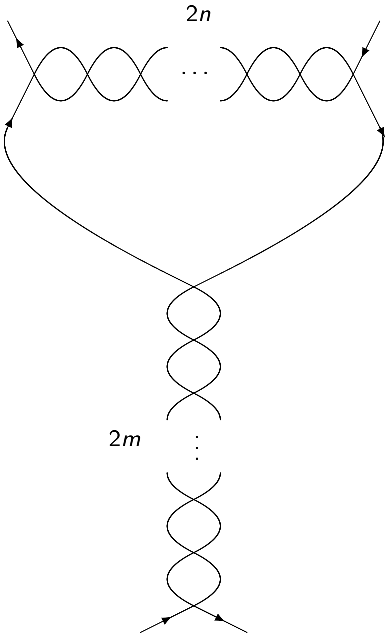
only for **defect zero**:

$$H_{[2]}^{\mathcal{K}} = 1 + [2]F_{[1]}^{\mathcal{K}}(q,A)\{Aq^2\}\{A/q\} + F_{[2]}^{\mathcal{K}^0}(q,A)\{Aq^3\}\{Aq^2\}\{A\}\{A/q\}$$

$$\text{defect} = \text{power}_{q^2}\left(AI_{[1]}^{\mathcal{K}}\right) \text{ - } 1$$

Double braids
have defect zero
and very special F:

$$H_{[r^s]}^{(m,n)} = \sum_{\lambda \subset [r^s]} D_{\tilde{\lambda}}(r) \cdot D_\lambda(s) \cdot Z_{r|s}^{\lambda} \cdot F_\lambda^{(m,n)}(q, A)$$


2n
2m

Differential expansion for double braids

$$H^{(m,n)}_{[r^s]} = \sum_{\lambda \in [r^s]} D_{\tilde{\lambda}}(r) \cdot D_{\lambda}(s) \cdot Z^{\lambda}_{r|s} \cdot F^{(m,n)}_{\lambda}(q,A)$$

$$\boxed{F^{(m,n)}_{\lambda}(q,A) \sim F^{(m)}_{\lambda}(q,A) \cdot F^{(n)}_{\lambda}(q,A)}$$

$$H_{[r^s]}^{(m,n)} = \sum_{\lambda \subset [r^s]} D_{\tilde{\lambda}}(r) \cdot D_\lambda(s) \cdot Z_{r|s}^{\lambda} \cdot \frac{F_\lambda^{(m)}(q,A) \cdot F_\lambda^{(n)}(q,A)}{F_\lambda^{(1)}(q,A) \cdot F_\lambda^{(-1)}(q,A)}$$

$$F_\lambda^{(m)}(q,A) = \sum_{\mu \in \lambda} f_{\lambda,\mu} \cdot \Lambda_\mu^{2m}$$

$$H_R^{(m,n)} = \sum_{\mu,\nu \in R \otimes \bar{R}} \frac{\sqrt{D_\mu D_\nu}}{D_R} \, \bar{S}_{\mu\nu} \, \Lambda_\mu^{2m} \Lambda_\nu^{2n}$$

$\bar{S}$ from DE for double braids

$$\bar{S}_{\mu\nu} = \frac{D_R}{\sqrt{D_\mu D_\nu}} \sum_{\mu,\nu\subset\lambda\subset R=[r^s]} \frac{D_{\tilde{\lambda}}(r)\cdot D_\lambda(s)\cdot Z^{\lambda}_{r|s}}{F^{(-1)}_\lambda(q,A)} \cdot f_{\lambda,\mu}\cdot f_{\lambda,\nu}$$

$$\bar{T}\bar{S}\bar{T} = ST^{-1}S^\dagger$$

$$S = ?$$

What are $f_{\lambda,\mu}$? $\Longleftarrow$ DE for twist knots $\mathrm{Tw}^{(m)}$

m		
...		
-3	8_1	
-2	6_1	
-1	4_1	figure eight
0		unknot
1	3_1	trefoil
2	5_2	
3	7_2	
...		

$$H^{(m)}_{[r^s]} = \sum_{\lambda\subset[r^s]} D_{\tilde{\lambda}}(r)\cdot D_\lambda(s)\cdot Z^{\lambda}_{r|s}\cdot F^{(m)}_\lambda(q,A)$$

$$F^{(m)}_\lambda(q,A) = \sum_{\mu\subset\lambda} f_{\lambda,\mu}\cdot\Lambda^{2m}_\mu$$

$F_\lambda^{(m)}(q,A) = \sum_{\mu\subset\lambda} f_{\lambda,\mu}\cdot\Lambda_\mu^{2m}$

$$F_{[1]} = \frac{1-A^{2m}}{1-A^2} = A\left(\frac{1}{\{A\}} - \frac{A^{2m}}{\{A\}}\right)$$

$$F_{[2]} = A\left(\frac{1}{\{Aq\}\{A\}} - [2]\frac{A^{2m}}{\{Aq^2\}\{A\}} + \frac{(qA)^{4m}}{\{Aq\}\{Aq^2\}}\right)$$

$$\frac{1}{[N]} + \,? \quad\longrightarrow\quad \frac{1}{[N]} - \frac{1}{[N]} = 0$$

$$\frac{1}{[N+1][N]} + \,? \quad\longrightarrow\quad \frac{1}{[N+1][N]} - \frac{[2]}{[N+2][N]} + \frac{1}{[N+2][N+1]}$$

$F_\lambda^{(m)}(q, A) = \sum_{\mu\subset\lambda} f_{\lambda,\mu} \cdot \Lambda_\mu^{2m}$

$$\prod_\lambda \frac{1}{[N + a'(\Box) - l'(\Box)]} + \; ? \;\longrightarrow\; \sum_{\mu\subset\lambda} \frac{\cdots}{\prod_{\lambda\cup\mu}[N + a'(\Box) - l'(\Box)]}$$

$F_\lambda^{(m)}(q,A) = \sum_{\mu\subset\lambda} f_{\lambda,\mu}\cdot\Lambda_\mu^{2m}$

$$\prod_\lambda \frac{1}{[N+a'(\Box)-l'(\Box)]} + ? \longrightarrow \sum_{\mu\subset\lambda} \frac{\dots}{\prod_{\lambda\cup\mu}[N+a'(\Box)-l'(\Box)]}$$

Numerator nor factorized, e.g.

$$f_{[3,2],[1]} \sim A^2q^8 + 2A^2q^6 + A^2q^4 + A^2q^2 - q^6 - q^4 - 2q^2 - 1$$

$F^{(m)}_\lambda(q,A) = \sum_{\mu\subset\lambda} f_{\lambda,\mu}\cdot\Lambda^{2m}_\mu \implies$ Skew Schur functions

$$f_{[3,2],[1]} \sim A^2q^8 + 2A^2q^6 + A^2q^4 + A^2q^2 - q^6 - q^4 - 2q^2 - 1$$

$$\sim \chi^*_{[3,2]/[1]}$$

$$\chi_\lambda\{p'_k + p''_k\} = \sum_{\mu\subset\lambda} \chi_{\lambda/\mu}\{p'_k\}\cdot\chi_\mu\{p''_k\}$$

$$p^*_k = \frac{\{A^k\}}{\{q^k\}} = \frac{[Nk]}{[k]}$$

$F_\lambda^{(m)}(q,A) = \sum_{\mu\subset\lambda} f_{\lambda,\mu}\cdot\Lambda_\mu^{2m} \implies$ Shifted skew Schur fns

$$f_{\lambda,\mu} \sim \left(\frac{\chi^*_{\lambda/\mu}\cdot\chi^*_\mu}{\chi^*_\lambda}\right)_{A\longrightarrow A\cdot q^{|\mu|-1}}$$

Proportionality coefficient is a fully factorized product of differentials

Explicitly known for $R=[rr]$ or $R=[2^s]$

Properties of HOMFLY

- Polynomiality and integralities
- Factorizations
- Equations
- Hurwitz integrability
- Vogel's universality (unification of E_8-sectors of all Lie algebras)
- Differential expansions
- ...

MANY THANKS

FOR YOUR ATTENTION!

MANY THANKS

to the ORGANIZERS!

VIVE l'AMITIÉ

Franco-Russe!