Matrix Models, Check-operators & Quantum Spectral Curves

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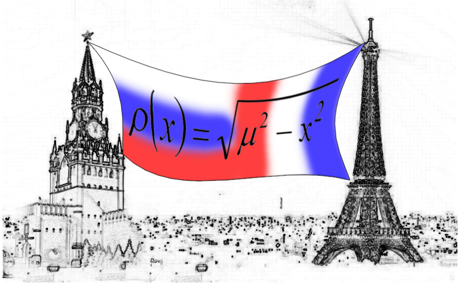

Plan

- Introduction
- Multiple solutions to the Virasoro constraints
- Check-operators
- Seiberg-Witten (SW) like solutions and integrable properties
- Quantum curves
- Quantum curves from degenerate conformal blocks
- Modular kernels in CFT

Old Story:

Matrix Models: Virasoro constraints = loop equations = Ward identities + integrability

New Story:

Check Operators: The space of solutions to the loop equations + quantized Whitham flows

An application:

Strings: matrix model networks as a tool to study topological strings and Nekrasov functions.
Algebraically: conformal blocks of Virasoro/W and Ding-Iohara-Miki algebras
Quantum field theory: supersymmetric quiver gauge theories of Seiberg-Witten type

Two-parametric deformations of Seiberg-Witten (SW) systems:

Gauge theory	Integrable system	Spectral curves
Nekrasov function	conformal matrix models, KP hierarchy	degenerate conf. block equation
	$\downarrow \epsilon_2 \to 0$	
quantum SW system	quantum many-body integrable system, $\hbar = \epsilon_1$	Schrödinger (Baxter) equation
	$\downarrow \epsilon_1 \to 0$	
Seiberg-Witten system	classical finite-dimensional integrable system	Spectral curve + Whitham flows

Matrix integrals and check-operators

Simplest example: the Hermitean matrix integral

$$Z = \int dM \exp\left[\ \mathrm{Tr} V(M)\ \right]$$

where we parameterize

$$V(M) = \sum_{k=0} t_k M^k$$

Loop equations = Virasoro constraints:

$$L_n Z = 0, \qquad n \geq -1$$

$$L_n = \sum t_k \frac{\partial}{\partial t_{k+n}} + \sum_{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b}$$

$$\frac{\partial Z}{\partial t_0} = NZ$$

Solutions as formal series

There is no solution such that Z is a power series!

Gaussian case ($t_2 \to t_2 - \alpha$):

$$Z = \int dM \exp\left[\, -\alpha \mathrm{Tr} M^2 + \mathrm{Tr} V(M)\,\right] = c_0 + c_1 t_1 + c_2^{(1)} t_1^2 + c_2^{(2)} t_2 + \dots$$

The coefficients are constructed from

$$\int dM \exp\left[\, -\alpha \mathrm{Tr} M^2\right] M^k$$

$$c_i \sim \alpha^{-i/2}$$

which is evident by dimensional argument.

More general (Dijkgraaf-Vafa) case:

$$Z = \int dM \exp\left[\ \mathrm{Tr}W(M) + \mathrm{Tr}V(M)\ \right]$$

$W(M) = \sum^n T_k M^k$, $t_k \longrightarrow T_k + t_k$ in the Virasoro constraints.
The coefficients are constructed from

$$Z = \int dM \exp\left[\ \mathrm{Tr}W(M)\ \right] M^k$$

i.e. there are combinations of T_k's in denominators.

How many solutions?

Solutions are parameterized by an arbitrary function of $n-2$ variables T_k. Two of them are fixed by

$$L_0 Z = 0, \qquad L_{-1} Z = 0$$

Thus, for $n = 2$ (Gaussian case) there is a unique solution.

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Technical tools: loop equations

Generating functions of correlators = resolvents:

$$\rho^{(1)}(z) = \left\langle \mathrm{Tr}\frac{1}{z-M} \right\rangle = \frac{1}{Z}\hat{\nabla}_z Z = \hat{\nabla}_z \mathcal{F}$$

$$\rho^{(2)}(z_1,z_2) = \left\langle \mathrm{Tr}\frac{1}{z_1-M}\mathrm{Tr}\frac{1}{z_2-M} \right\rangle_c = \hat{\nabla}_{z_1}\hat{\nabla}_{z_2}\mathcal{F}$$

$$\ldots$$

where

$$\hat{\nabla}_z = \sum_k \frac{1}{z^{k+1}}\frac{\partial}{\partial t_k}, \qquad Z = \exp\mathcal{F}$$

Loop equations:

$$[T(z)Z]_- = 0, \qquad T(z) \equiv \sum \frac{L_n}{z^{n+2}}$$

$$\rho^{(1)}(z)^2 + \hat{\nabla}_z\rho^{(1)}(z) + W'(z)\rho^{(1)}(z) + \underbrace{\Big[W'(z)\rho^{(1)}(z)\Big]_+}_{\text{polynomial of degree } n-2} + \underbrace{\Big[V'(z)\rho^{(1)}(z)\Big]_-}_{=0 \text{ as } t_k\to 0} = 0$$

Check-operator: acting on the space of solutions.

$$f_{n-2}(z) = \Big[W'(z)\rho^{(1)}(z)\Big]_+ \equiv \check{R}_z\mathcal{F}, \qquad \check{R}_z = -\sum_{a,b}(a+b+2)T_{a+b+2}z^a\frac{\partial}{\partial T_b}$$

Classical spectral curve:

Genus expansion:

$$t_k, T_k \to \frac{1}{g}t_k, \frac{1}{g}T_k, \qquad Z = \exp\left(\frac{1}{g^2}\mathcal{F}\right), \qquad \mathcal{F} = \sum_k g^{2k}\mathcal{F}_k$$

Leading term of the genus expansion:

$$\rho^{(1)}(z) = \frac{-W'(z) + y(z)}{2}$$

where

$$y(z)^2 \equiv W'(z)^2 - 4f_{n-2}(z)$$

determines the classical spectral curve.

Examples:

- Gaussian case: $n = 2$, $f_0(z) = const$, $y^2 = z^2 - const$: semi-circle distribution.
- W_3-case: $n = 3$, $f_1(z)$ is a linear function, the spectral curve is a torus, the space of solutions is described by a function of one variable.
  Meaning: typical integration in the matrix (eigenvalue) integral

$$\int dx e^{W_3(x)}$$

  implies two independent choices of contour (Airy functions). N eigenvalues in the integrand part into two groups (two contours), N_1 and N_2, $N_1 + N_2 = N$. This describes the two-cut solution (torus). Arbitrary function of one variable corresponds to one arbitrary variable, fraction N_1/N_2.

Summary of general properties:

1. Any solution is unambiguously labeled by an arbitrary function of $n-2$ T-variables: the *bare* $\mathcal{F}^{(0)}(T)$.

2. Solutions of the Virasoro constraints (or loop equations) are constructed from $\mathcal{F}^{(0)}(T)$ by an evolution operator $\hat{U}(T,t)$ that does not depend on $\mathcal{F}^{(0)}(T)$

:

$$Z(T,t) = \hat{U}(T,t)e^{\mathcal{F}^{(0)}(T)}$$

3. The evolution operator can be completely expressed in terms of the unique operator

$$\check{y} \equiv \sqrt{W'(x)^2 - 4\check{R}(x)}, \qquad \check{R}(x) \equiv -\sum_{a,b=0}(a+b+2)T_{a+b+2}x^a\frac{\partial}{\partial T_b}$$

its derivatives and $W'(x)$.

Consequence:

$$\rho^{(1)}(z) = \hat{\nabla}_z(t)\mathcal{F} = \check{\nabla}_z(T)\mathcal{F}$$

The main check-operator $\check{\nabla}_z$ is expressed through y, its derivatives and $W'(x)$. Important: $[\hat{\nabla}_{z_1}, \hat{\nabla}_{z_2}] = 0$, but $[\check{\nabla}_{z_1}, \check{\nabla}_{z_2}] \neq 0$.

Main property:

$$\boxed{[\oint_{A_i} dz\check{\nabla}_z, \oint_{B_j} dz\check{\nabla}_z] = \delta_{ij}}$$

The curve: $y^2 = W'^2(x) - 4\check{R}(x)\mathcal{F}^{(0)}$.

DV/SW system:

Choose the basis of eigenfunctions $\oint_{A_i} dz\check{\nabla}_z Z_a = a_i Z_a$, i.e. $\oint_{A_i} dz\check{\nabla}_z \mathcal{F}_a = \oint_{A_i} dz\rho^{(1)}(z) = a_i$, then

$$\oint_{B_i} dz\rho^{(1)}(z) = \frac{\partial \mathcal{F}}{\partial a_i}$$

N_i are associated with

$$a_i = \oint_{A_i} \rho^{(1)}(z)dz$$

Integrable properties:

- $Z_N(t)$ is a τ-function of the Toda chain (as a formal series)
- $Z_{DV}(T, N_i)$ is the SW system; it satisfies the Whitham hierarchy in the planar limit, T_i are Whitham flows.

Quantum spectral curve:

The Backer-Akhiezer function:

$$\Psi_{BA}(z) = e^{V(z)/2}\Psi(z)$$

where

$$\Psi(z) = \frac{Z\left(t_k - \frac{1}{kz^k}\right)}{Z(t)} = \frac{1}{Z(t)} z^N e^{\int^z d\xi \hat{\nabla}_\xi} = \langle \det(z_M) \rangle$$

and < ... > means the matrix model average. From the Virasoro constraints the quantum spectral curve:

$$\boxed{\left[\partial_z^2 + V'(z)\partial_z + \check{R}_z\right]\Psi(z) = 0}$$

In the limit, $\partial \log \Psi(z) = \rho^{(1)}(z)$ it turns into the classical spectral curve (planar loop equation):

$$\rho^{(1)}(z)^2 + V'(z)\rho^{(1)}(z) + \check{R}_z\mathcal{F} = 0$$

The equation for the Baker-Akhiezer function looks as

$$\left[\partial_z^2 - \frac{1}{2}V''(z) + \frac{1}{4}V'(z)^2 - \frac{1}{2}[\check{R}_z V(z)] + \check{R}_z\right]\Psi_{BA}(z) = 0$$

$\check{R}_z$ contains the derivatives w.r.t. the Whitham times.

AGT and conformal blocks: quantum spectral curve

Conformal block:

$G(x,\Delta;\Delta_i,c)$, $\Delta=(Q-\alpha)\alpha$, $c=1+6Q^2$, $Q=b-1/b$, $V_\alpha(z)=e^{i\alpha\phi(z)}$.

Degenerate field:

$(b^2L_{-1}^2-L_{-2})V_{1/2b}(z)$ is a primary field, i.e. $V_{1/2b}(z)$ is degenerate at the second level. The equation for the 5-point block with the degenerate field at z:

$$\left[b^2z(z-1)\partial_z^2+(2z-1)\partial_z-\underbrace{\frac{q(q-1)}{z-q}\partial_q+\text{rational function of } q}_{\text{check-operator}}\right]G_5(z|0,q,1,\infty)=0$$

This is the quantum spectral curve, q is a counterpart of T_k.

Comment.

In the limit when all $\Delta_i \to \infty$, this equation is reduced to the non-stationary Schrödinger $SU(2)$ periodic Toda chain equation:

$$\left(\partial_z^2 - 2\Lambda^2 \cosh z + \frac{1}{4}\frac{\partial}{\partial\Lambda}\right) G_5^{Toda} = 0$$

where Λ is the limit of rescaled q.
$\log \Lambda$ is known to play the role of the first Whitham time in the Seiberg-Witten theory.

Conformal matrix model.

This quantum curve is for the conformal matrix model:

$$G_4(0,q,1,\infty) = q^{2\alpha_1\alpha_2}(1-q)^{2\alpha_2\alpha_3}\int\prod_i du_i\Delta^{2b^2}(u)u_i^{2b\alpha_1}(1-u_i)^{2b\alpha_3}(q-u_i)^{2b\alpha_2}$$

where $\Delta(u)$ is the Van-der-Monde determinant and α, α_4 are fixed from the conditions:

- N_1 contours $[0,q]$ with
$$bN_1 = \alpha - \alpha_1 - \alpha_2$$
- N_2 contours $[0,1]$ with
$$bN_2 = Q - \alpha - \alpha_3 - \alpha_4$$

N_1 and N_2 are associated with the Dijkgraaf-Vafa N_i. Since

$$G_4 = \left\langle V_{\alpha_1}(0)V_{\alpha_2}(q)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\left(\int_0^q V_b(u)du\right)^{N_1}\left(\int_0^1 V_b(u)du\right)^{N_2}\right\rangle_{CFT}$$

the degenerate block $G_5 = \langle V_{1/2b}(z)\dots\rangle_{CFT}$, $< V_{1/2b}(z)V_b(u) >_{CFT} = z-u$, $G_5 = < \det(z-u_i) >$,

the equation for G_5 is exactly the quantum spectral curve.

Modular kernels in CFT

Modular kernel

$$G(x,a;a_i,b) = \int da' K(a,a';a_i,b) G(1-x,a';a_i,b)$$

$a_i = \alpha_i - Q/2.$

Modular kernel (Ponsot, Teschner, 1999)

$$K(a,a';a_i,b) \sim \int dx \prod_{i=1}^{4} \frac{S_b(\xi_i)}{S_b(\zeta_i)}$$

ξ_i, ζ_i are linear functions of all parameters and x.

Representation of $G(x,a;a_i,b)$ as a β-ensemble with $\beta = b^2$

$$K(a,a';a_i,b) = e^{4\pi i a a'}$$

1-point toric conformal block

$$G(\tau,a;\mu)=1+q\Big(\frac{\Delta_{ext}(1-\Delta_{ext})}{2\Delta}+1\Big)+O(q^2)$$

with $\Delta_{ext}=\mu(Q-\mu)$, μ is the adjoint hypermultiplet mass.

$$G(\tau,a;\mu)=\int da' K(a,a';\mu)G(-\tau^{-1},a';\mu)$$

Modular kernel due to Ponsot, Teschner:

$$K(a,a';\mu)=\int d\xi \frac{S_b(\xi+\mu/2-a')S_b(\xi+\mu/2+a')}{S_b(\xi+Q-\mu/2-a')S_b(\xi+Q-\mu/2+a')}e^{4\pi i a\xi}$$

Modular kernel from β-ensemble:

$$G(\tau,a;\mu)=\frac{1}{N(a)}Z(\tau,a;\mu), \qquad N(a)=\frac{\Gamma_b(2a+\mu)\Gamma_b(2a+Q-\mu)}{\Gamma_b(2a)\Gamma_b(2a+Q)}$$

$$Z(\tau,a;\mu)=\int da' e^{2\pi i a a'}Z(-\tau^{-1},a';\mu)$$

Quantum oscillator

$$A = e^{i\hat{P}}, \qquad B = e^{i\hat{Q}}$$

$$\hat{A} Z_a(Q) = e^{ia} Z_a(Q), \qquad \hat{B} \tilde{Z}_a(Q) = e^{ia'} \tilde{Z}_{a'}(Q)$$

$$Z_a(Q) = \int da' e^{iaa'/\hbar} \tilde{Z}_{a'}(Q)$$

Check-operators

$$\check{A} = e^{ia}, \qquad \check{B} = e^{\hbar \partial_a}$$

$$\check{A}_a K(a, a') = \check{B}_{a'} K(a, a')$$

We expect the conformal block to be an eigenfunction of some $\mathcal{L}_A$:

$$\mathcal{L}_A G = \lambda G, \qquad \mathcal{L}_B G = \Lambda(\partial_\lambda) G$$

Claim

$$\mathcal{L}_\gamma = e^{b \oint_\gamma \check{\nabla}}$$

Since $[\mathcal{L}_A, \mathcal{L}_B] = 1$, we obtain that $K(a, a'; \mu)$ is Fourier!!!

Subtlety

$\check{\nabla}$ has two branches, i.e. there are $\check{\nabla}^{\pm}$! G is globally defined but $Z(a)$ is *not*! There are two branches at $a > 0$ and $a < 0$. Thus,

$$\mathcal{L}_\gamma = \left[\frac{1}{N(a)} e^{b \oint_\gamma \check{\nabla}^+} N(a) + \frac{1}{N(-a)} e^{-b \oint_\gamma \check{\nabla}^-} N(-a) \right]$$

One can realize check-operators in the space of eigenvalues:

$$\oint_A dz\check{\nabla}_z^{\pm} \to \pm 2\pi i a, \qquad \oint_B dz\check{\nabla}_z^{\pm} \to \pm\frac{1}{2}\partial_a$$

Thus, one obtains

$$\mathcal{L}_B = \frac{\Gamma(2ab)\Gamma(bQ+2ab)}{\Gamma(b\mu+2ab)\Gamma(b(Q-\mu)+2ab)} e^{\frac{b}{2}\partial_a} + (a \to -a)$$

Since $\mathcal{L}'_A = \cos 2\pi ba$ the equations $\check{A}_a K(a,a') = \check{B}_{a'} K(a,a')$ for the modular kernel becomes

$$\frac{1}{2}\left(\frac{\sin 2\pi b(a-\mu/2)}{\sin 2\pi ba} e^{-\frac{b}{2}\partial_a} + \frac{\sin 2\pi b(a+\mu/2)}{\sin 2\pi ba} e^{\frac{b}{2}\partial_a}\right) K(a,a') = \cos 2\pi ba' \; K(a,a')$$

At large a only one exponential survives giving the pure exponential kernel. The solution of the full equation is given by

$$K(a,a';\mu) = \int d\xi C_1(\xi) C_2(a') \frac{S_b(\xi+\mu/2-a')S_b(\xi+\mu/2+a')}{S_b(\xi+Q-\mu/2-a')S_b(\xi+Q-\mu/2+a')} e^{4\pi i a\xi}$$

which coincides with the Ponsot-Teschner formula at $C_1 = C_2 = 1$.

Thank you for your attention!