SUSY Gauge Theories, W-algebras and Isomonodromic Deformations

Andrei Marshakov
Lebedev, ITEP, NRU HSE & CAS Skoltech

Random Geometry and Physics
Institute Henri Poincaré, Paris

October 2016

based on:
P.Gavrylenko & AM: JHEP05 (2014) 097;
P.Gavrylenko & AM: JHEP02 (2016) 181;
M.Bershtein, P.Gavrylenko & AM: to appear.

Emerging of the Seiberg-Witten geometry: extremal geometry for Nekrasov partition function

$$\mathcal{Z}(\{\mathbf{a}\},\{q\},\{\mathbf{m}\}|\epsilon_{1,2}) = \sum_{\{\mathbf{Y}\}} \prod_{\Phi} Z_{\Phi}(\{\mathbf{Y}\},\{\mathbf{a}\},\{q\},\{\mathbf{m}\}|\epsilon_{1,2}) \sim$$
$$\sim \exp\left(\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\{\mathbf{a}\},\{q\},\{\mathbf{m}\}) + \ldots\right) \tag{1}$$

($\prod_{\Phi}$ - product over multiplets of a 4d SUSY gauge theory).

Another correspondence (like for "ITEP matrix models" - forgetting original geometry)

$$\mathcal{Z}(\{\mathbf{a}\},\{q\},\{\mathbf{m}\}|\epsilon_{1,2}) = \mathcal{G}(\{\mathbf{a}\},\{q\},\{\mathbf{m}\}|\epsilon_{1,2}) \tag{2}$$

with conformal blocks in 2d conformal theory.

A particular case

$$\mathcal{G}(\{\mathbf{a}\},\{q\},\{\mathbf{m}\}|\epsilon_{1,2})\Big|_{\epsilon_1=-\epsilon_2=\hbar,\{\mathbf{m}\}\sim\hbar} = \\ = \exp\left(-\frac{1}{\hbar^2}\mathcal{F}(\{\mathbf{a}\},\{q\})\right)\tau_B(\{q\}) \tag{3}$$

of the exact conformal blocks for W-algebras.

Goes “beyond” AGT correspondence (like ITEP matrix mod-elsgo beyond original integrals over random matrices)!

Conformal block - *not* in original BPZ normalization, absorbes 3-point functions or structure constants.

Seiberg-Witten theory: a curve Σ with Krichever's data (roughly dz and dx or $dS = xdz$)

$$\frac{\partial^2 \mathcal{F}}{\partial a_I \partial a_J} = \mathcal{T}_{IJ}(\Sigma), \qquad \text{or } a_I = \oint_{A_I} dS \text{ and } \frac{\partial \mathcal{F}}{\partial a_I} = \oint_{B_I} dS$$
$$\frac{\partial}{\partial q_\alpha}\mathcal{F} = \tfrac{1}{2} \sum_{\pi(q_\alpha^i)=q_\alpha} \text{Res}_{\ q_\alpha^i} \frac{(dS)^2}{dz} \tag{4}$$

where the second formula also implies $\pi : \Sigma \to \Sigma_0$ (physics $\Sigma_0(\{q\}) = \Sigma_{UV}$. SUSY gauge theory in 4d (5d etc).

Exact conformal blocks (*not* a Nekrasov's solution)

$$\mathcal{F} = \tfrac{1}{2} \sum_{I,J} a_I \mathcal{T}_{IJ}(\{q\}) a_J + \sum_I a_I U_I(\{q\}) + \tfrac{1}{2} Q(\{q\}) \tag{5}$$

is just a *quadratic form* (supplemented by the Rauch formulas). Constructed in terms of representation theory for 2d conformal theories.

Exact conformal blocks (from Al.Zamolodchikov, 80-s):

- Twist fields at $c = 1$ Virasoro theory:

$$\langle \mathcal{O}(\infty)\mathcal{O}(1)|_a \mathcal{O}(q)\mathcal{O}(0)\rangle = \frac{16\exp(i\pi\tau a^2)}{q^{1/8}(1-q)^{1/8}\theta(0|\tau)} \tag{6}$$

- $\mathcal{O}(z)$ is Virasoro primary with $\Delta = 1/16$, corresponding $|\mathcal{O}\rangle$ is an eigenstate of L_0, but *not* of a current, unlike intermediate state with fixed charge a;

- Quadratic in a expression (prepotential in conformal 4d SUSY gauge theory), where $\tau = \frac{\oint_B \frac{dx}{y}}{\oint_A \frac{dx}{y}}$ - period "matrix" of torus $y^2 = x(x-q)(x-1)$.

Theories with W-symmetry (with integer central charges), an example of "well-defined" conformal blocks, but *not* given by Nekrasov functions. New (?) formulas for the characters ... lattice theta-functions.

Isomonodromy/CFT correspondence or *Kiev formula* (GIL & Gavrylenko)

$$\tau_{IM}(q;\ldots) = \sum_{\boldsymbol{n}\in\Gamma(\mathfrak{sl}_N)} e^{2\pi i(\mathbf{n},\mathbf{b})}\cdot\mathcal{G}(\mathbf{a}+\mathbf{n},q,\{\mathbf{m}\}|\ldots) \tag{7}$$

Twist fields = quasi-permutation monodromies:

$$\begin{aligned}\tau_{IM}(q|\mathbf{a},\mathbf{b}) = \sum_{\mathbf{n}\in\mathbb{Z}^g}\mathcal{G}(q|\mathbf{a}+\mathbf{n})e^{2\pi i(\mathbf{n},\mathbf{b})} = \\ = \tau_B(\boldsymbol{q})\Theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\Omega)\end{aligned} \tag{8}$$

Twist field for Virasoro algebra at $c = 1$, $\mathcal{O}(z)$

$$I(z)\mathcal{O}(w) \sim (z-w)^{-\frac{1}{2}}(\alpha_{-\frac{1}{2}}\mathcal{O})(w) \tag{9}$$

corresponds to the only nontrivial element $M \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $\mathtt{W}(\mathfrak{sl}_2)$. In the vicinity of twist field

$$I(z) = \sum_{r\in\mathbb{Z}+1/2} \frac{\alpha_r}{z^{r+1}} \tag{10}$$

and for the character of an irreducible current module $\mathcal{H}_{\mathcal{O}} \ni \{\alpha_{-r_1}\ldots\alpha_{-r_k}|\mathcal{O}\rangle; L_0|\mathcal{O}\rangle = \frac{1}{16}|\mathcal{O}\rangle\}$

$$\chi_{\mathcal{O}}(q) = \mathrm{Tr}_{\mathcal{F}_{\mathcal{O}}} q^{L_0} = \frac{q^{\frac{1}{16}}}{\prod_{r>0}(1-q^r)} = \frac{\sum_{r>0} q^{r^2/4}}{\prod_{n>0}(1-q^n)} \tag{11}$$

on gets the "Gauss-Zamolodchikov" formula.

Gauss:

$$\frac{\prod_{n>0}(1-q^n)}{\prod_{r>0}(1-q^r)} = \prod_{n>0}(1-q^{2n}) \prod_{r>0}(1+q^r) = \\
= \prod_{n>0}(1-q^n) \prod_{n>0}(1+q^n) \prod_{r>0}(1+q^r) = \\
= \prod_{n>0}(1-q^{n/2}) \prod_{n>0}(1+q^{n/2})^2 = \\
= \sum_{s>0} q^{s^2/4-1/16} = \theta(\tau/4|2\tau) \tag{12}$$

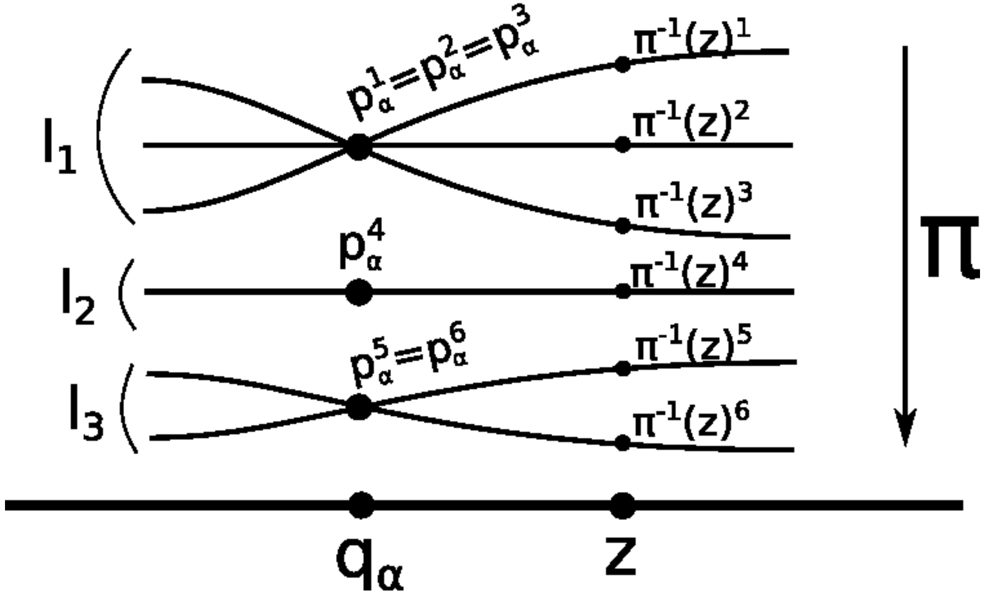
just using a simple identity and Jacobi triple product formula.

Zamolodchikov: expansion of twisted Fock module

$$\chi_{\mathcal{O}}(q) = \sum_{r>0} \mathcal{X}_{r^2/4}(q) \tag{13}$$

over the irreducible Virasoro representations.

The (quasi-)permutation or twist fields $\mathcal{O}$ for $\mathfrak{g} = \mathfrak{gl}(N)$ (an arbitrary N-sheet cover)



- Primary fields of the W-algebra (and *not* of the current algebra!) - *non* degenerate in terms of W-representation theory;

- Their quantum numbers $\mathcal{O}_{s,\mathbf{r}}(z) \mapsto |s,\mathbf{r}\rangle$ are symmetric functions of the Weyl vector (for each cycle in $s = [l_1, \ldots, l_k]$), e.g. $\Delta = \frac{1}{2}\mathrm{Tr}(\log M_\gamma)^2$ or

$$\Delta = \frac{1}{2}\sum\left(\frac{\log \lambda_{i,v_i}}{2\pi i}\right)^2 = \sum_{i=1}^{k} \frac{l_i^2 - 1}{24 l_i} + \sum_{i=1}^{k} \frac{1}{2} l_i r_i^2 \qquad (14)$$

- Consistent with geometric formula for $t = \langle T \rangle$

$$t_z(p) = t_\xi(p)\left(\frac{d\xi}{dz}\right)^2 + \frac{1}{12}\{\xi, z\} \qquad (15)$$

which gives the dimensions, due to $\frac{1}{12}\{\xi, z\} = \frac{l^2-1}{24 l^2}\frac{1}{z^2}$ for $z = \xi^l$.

The "irreducible" case: the Coxeter element $\in \mathtt{W}$, so that the W-charges are symmetric functions of

$$\mathbf{a} \sim \frac{1}{N}\rho \tag{16}$$

For generic series of Lie algebras $\mathfrak{g}$ shoud be replaced by

$$\begin{gathered} \mathbf{a} \sim \frac{1}{h}\rho^{\vee} \\ (\rho^{\vee}, \alpha) = 1, \qquad \alpha \in \Pi \end{gathered} \tag{17}$$

(mod h for affine roots).

Are the analogs of the "Gauss-Zamolodchikov" formula?

Direct generalization of the Gauss-Zamolodchikov formula for $\mathcal{O}=\mathcal{O}_{\rho/N}$:

$$\chi_{\mathcal{O}}(q)=\frac{q^{\frac{N^2-1}{24N}}}{\prod\limits_{k=1}^{\infty}(1-q^{k/N})}=q^{\frac{N^2-1}{24N}}\frac{\Theta_N(\tau\rho/2N|q)}{\prod\limits_{k=1}^{\infty}(1-q^k)^N}= \\ =\sum_{\alpha\in\Gamma(\mathfrak{sl}_N)}\mathcal{X}_\alpha(q) \tag{18}$$

i.e. the character of N twist boson's module is expanded over the W-characters, and since (logarithm up to $\sum n_k=0$)

$$\Delta_{\mathbf{n}}=\tfrac{1}{2}\sum_k(a_k+n_k)^2=\frac{N^2-1}{24N}+\tfrac{1}{2}\mathbf{n}^2+\frac{1}{N}\rho\cdot\mathbf{n} \tag{19}$$

the sum is taken over the root *lattice* ($\mathbf{n}\in\mathbb{Z}^N$, $\sum_j n_j=0$).

Product formulas for the *lattice* theta-functions

$$\sum_{\alpha\in\Gamma(\mathfrak{sl}_N)} q^{\frac{1}{2}(\alpha+\frac{\rho}{N},\alpha+\frac{\rho}{N})} = q^{\frac{N^2-1}{24N}}\Theta_N\left(\tau\rho/2N|q\right) =$$

$$= q^{\frac{N^2-1}{24N}}\frac{\prod_{k=1}^{\infty}(1-q^k)^N}{\prod_{k=1}^{\infty}(1-q^{k/N})} = q^{\frac{N^2-1}{24N}}\frac{\prod_{k=1}^{\infty}(1-q^k)^{N-1}}{\prod_{l=1}^{N-1}\prod_{k=1}^{\infty}(1-q^{k+l/N})} \tag{20}$$

or in terms of Dedekind functions $\eta(q) = q^{1/24}\prod_{k=1}^{\infty}(1-q^k)$ just

$$\sum_{\alpha\in\Gamma(\mathfrak{sl}_N)} q^{\frac{1}{2}(\alpha+\frac{\rho}{N},\alpha+\frac{\rho}{N})} = \frac{\eta(q)^N}{\eta(q^{1/N})} \tag{21}$$

- Representation theory of $\widehat{\mathfrak{gl}(N)}_1$

$$\chi_{\mathcal{O}}(q) \sim \mathrm{Tr} q^{L_0 + \frac{1}{h}(\rho^\vee, H)} \tag{22}$$

  Free fermions (again line in "ITEP matrix model"). Lepowsky-Wilson bosons from *twisted* fermions.

- Generalization for other series: product formulas for the lattice theta-functions. For $D_r = \mathfrak{so}_{2r}$ and $B_r = \mathfrak{so}_{2r+1}$ this come from real (even and odd number of) fermions.

$W_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{gl}(N)$, $W_N \oplus H \subset U(\widehat{\mathfrak{gl}(N)}_1)$ with integer Virasoro central charge

$$c = \left.\frac{K \,\mathrm{dim}\, \mathfrak{g}}{K + C_V}\right|_{K=1} + 1 = \frac{N^2 - 1}{1 + N} = (N - 1) + 1 = N \qquad (23)$$

Fermionic representation ($\alpha, \beta = 1, \ldots, N$)

$$J_{\alpha\beta}(z) = \Big(\tilde{\psi}_\alpha(z)\psi_\beta(z)\Big) =: \tilde{\psi}_\alpha(z)\psi_\beta(z) : + \delta_{\alpha\beta}\frac{\sigma_\alpha}{z} \qquad (24)$$

Representation for the generators: fermionic

$$\sum_{\alpha=1}^{N} \tilde{\psi}_\alpha(z + \frac{t}{2})\psi_\alpha(z - \frac{t}{2}) = \frac{N}{t} + \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} U_k(z) \qquad (25)$$

or bosonic (in terms of Cartan currents $J_\alpha = J_{\alpha\alpha}(z) \in \mathfrak{h}$)

$$W_k(z) = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_k} : J_{\alpha_1}(z) J_{\alpha_2}(z) \ldots J_{\alpha_k}(z) : \qquad (26)$$
$$k = 1, \ldots, N$$

twisted fermions $\rightarrow$ twisted bosons $\rightarrow$ characters in principal specialization

$$q^{-\Lambda}\chi(\mathsf{L}_{\Lambda}) = \prod_{\alpha^{\vee}\in\Delta_{+}^{\vee}} \left(\frac{1 - q^{\frac{1}{h}(\Lambda+\rho,\alpha^{\vee})}}{1 - q^{\frac{1}{h}(\rho,\alpha^{\vee})}} \right)^{\mathrm{mult}(\alpha^{\vee})} \tag{27}$$

for an integrable module with the highest weight Λ.

For the non-simply laced case (B-series) one has to take product here over the root lattice of twisted Kac-Moody algebra $\mathfrak{g}^{\vee} = A_{2r-1}^{(2)}$.

The simply-laced generalization for any $\mathfrak{g}$ of ADE-type

$$\sum_{\alpha\in\Gamma_{\mathfrak{g}}} q^{\frac{1}{2}(\alpha+\frac{\rho}{h},\alpha+\frac{\rho}{h})} = q^{\frac{\dim\mathfrak{g}}{24h}}\Theta_{\mathfrak{g}}(\tau\rho/h|q) = \\ = \eta(q)^r \cdot q^{r/24h}\prod_{n>0} P_W(q^{n/h}) \tag{28}$$

where $r = \operatorname{rank}\mathfrak{g}$, $h = h_{\mathfrak{g}}$ is the Coxeter number and P_W is the characteristic polynomial of the Coxeter element of the Weyl group $W = W_{\mathfrak{g}}$.

For A_r-series, with $r = N-1$, $h = N$ and $\dim\mathfrak{sl}_N = N^2-1$, using that

$$P_{W(\mathfrak{sl}_N)}(x) = \frac{1-x^N}{1-x} \tag{29}$$

one gets back to our formula.

For the D_r-series, with $h = 2(r-1)$ and $\dim \mathfrak{so}_{2r} = r(2r-1)$, one takes

$$P_{W(\mathfrak{so}_{2r})}(x) = (1+x)(1+x^{r-1}) = \frac{1-x^2}{1-x}\frac{1-x^{2(r-1)}}{1-x^{r-1}} \tag{30}$$

and gets

$$\begin{aligned} \sum_{\alpha \in \Gamma_{\mathfrak{so}_{2r}}} q^{\frac{1}{2}(\alpha+\frac{\rho}{h},\alpha+\frac{\rho}{h})} = q^{\frac{r(2r-1)}{48(r-1)}} \Theta_{\mathfrak{so}_{2r}}(\tau\rho/h|q) = \\ = \frac{\eta(q)^{r+1}\eta(q^{1/(r-1)})}{\eta(q^{1/2})\eta(q^{1/2(r-1)})} \end{aligned} \tag{31}$$

A verification check for $r = 3$ ...

Similarly, for $\mathfrak{g} = \mathfrak{so}_{2r+1}$

$$\frac{\prod_{k\geq 0}(1+q^{k+\frac{1}{2}})}{\prod_{n>0}(1-q^n)^r} \sum_{\alpha\in\Gamma(\mathfrak{so}_{2r+1})} q^{\frac{1}{2}(\alpha,\alpha)+\frac{1}{2r}(\rho^\vee,\alpha)} =$$

$$= 2\prod_{n>0}(1+q^n)\prod_{j=1}^{r}\frac{1}{\prod_{k\geq 0}(1-q^{k+\frac{2j-1}{2r}})} = \tag{32}$$

$$= \frac{2\prod_{n>0}(1+q^n)}{\prod_{k\geq 0}(1-q^{\frac{2k+1}{2r}})}$$

The r.h.s. has an obvious sense of the trace

$$\mathrm{Tr}_{\mathcal{F}_\psi^{\otimes(2r+1)}}\, q^{L_0+\frac{1}{2r}(\rho^\vee,H)} = \mathrm{Tr}_{\mathcal{V}_0\oplus\mathcal{V}_1}\, q^{L_0+\frac{1}{2r}(\rho^\vee,H)} =$$
$$= 2\mathrm{Tr}_{\mathcal{V}_0}\, q^{L_0+\frac{1}{2r}(\rho^\vee,H)} = 2\mathrm{Tr}_{\mathcal{V}_1}\, q^{L_0+\frac{1}{2r}(\rho^\vee,H)} \tag{33}$$

Similarly, for a spinor representation

$$\mathrm{Tr}_{\mathcal{V}_S}\; q^{L_0+\frac{1}{2r}(\rho^\vee,H)} = \frac{\prod_{k\geq 0}(1+q^{k+1/2})}{\prod_{k\geq 0}(1-q^{\frac{2k+1}{2r}})} = \tag{34}$$
$$= \frac{\prod_{n>0}(1+q^n)}{\prod_{n>0}(1-q^n)^r} \sum_{\alpha\in\Gamma(\mathfrak{so}_{2r+1})+\frac{1}{2}\sum_j e_j} q^{\frac{1}{2}(\alpha,\alpha)+\frac{1}{2r}(\rho^\vee,\alpha)}$$

"Emerging of a fermion" from the Weyl-Kac type formulas for the characters.

A generic product formula for the lattice theta-functions (?)

$$\Theta_{\mathfrak{g}}\left(\tau\rho^{\vee}/h|q\right) \sim \sum_{\alpha\in\Gamma(\mathfrak{g})} q^{\frac{1}{2}(\alpha,\alpha)+\frac{1}{h}(\rho^{\vee},\alpha)} \tag{35}$$

More open questions:

- W-algebras for non simply-laced $\mathfrak{g}$;
- Classification of the twist-field representations (by elements of the Weyl group);
- Isomonodromic problem for other series ...