Form factors in $\mathfrak{gl}(2|1)$-invariant integrable models

A. Hutsalyuk, **A. L.**, S. Z. Pakuliak, E. Ragoucy, N. A. Slavnov

October 20, 2016

Table of contents

- Some definitions
- Bethe vectors
- Scalar product
- Form factors

Yang-Baxter equation

$$R_{12}(x,y)R_{13}(x,z)R_{23}(y,z) = R_{23}(y,z)R_{13}(x,z)R_{12}(x,y)$$

Yang-Baxter equation holds in the tensor product of the 3 spaces $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1}$.
Notation $\mathbb{C}^{2|1}$ means that algebra is the $\mathbb{Z}_2$-graded vector space with the grading $[1] = [2] = 0$, $[3] = 1$.
The rational solution is

$$R(x,y) = \mathbb{I} + g(x,y)\mathbb{P}, \qquad \mathbb{P} = \sum_{a,b} (-1)^{[b]} \mathbf{e}_{ab} \otimes \mathbf{e}_{ba},$$

where $g(x,y) = \frac{c}{x-y}$, and c is a constant.

From RTT-relation

$$R(u,v)\big(T(u)\otimes\mathbb{I}\big)\big(\mathbb{I}\otimes T(v)\big)=\big(\mathbb{I}\otimes T(v)\big)\big(T(u)\otimes\mathbb{I}\big)R(u,v).$$

we obtain commutation relations in the graded algebras

$$[T_{ij}(u),T_{kl}(v)\}=$$
$$=(-1)^{[i]([k]+[l])+[k][l]}g(u,v)\,[T_{kj}(v)T_{il}(u)-T_{kj}(u)T_{il}(v)]\,,$$

where $[,\}$ is a graded commutator.
The changing in the commutation relations drastically changes the further results.

Useful notation

$$f(x,y)=\frac{x-y+c}{x-y},\qquad g(x,y)=\frac{c}{x-y},$$

$$h(x,y)=\frac{x-y+c}{c},\qquad t(x,y)=\frac{c^2}{(x-y+c)(x-y)}.$$

the shorthand notation for sets and products

$$\bar{u}=\{u_1,u_2,\dots,u_n\},\qquad \bar{u}_j=\bar{u}\setminus u_j.$$

$$h(\bar{u},v)=\prod_{u_j\in\bar{u}}h(u_j,v);\quad g(v,\bar{u}_\ell)=\prod_{\substack{u_j\in\bar{u}\\ u_j\neq u_\ell}}g(v,u_j).$$

It follows from the commutation relations that

$$h(u,v)\,T_{i3}(u)\,T_{i3}(v)=h(v,u)\,T_{i3}(v)\,T_{i3}(u).$$

Then, notation

$$T_{j3}(\bar{u})=\prod_{k=1}^{n}T_{j3}(u_k)$$

is not well defined. Introduce instead $\mathbb{T}_{13}$, $\mathbb{T}_{23}$ as follows

$$\mathbb{T}_{j3}(\bar{u})=\frac{T_{j3}(u_1)\dots T_{j3}(u_n)}{\prod_{n\geq\ell>m\geq1}h(u_\ell,u_m)}.$$

Bethe vector

Bethe vector in the $\mathfrak{gl}(2|1)$ symmetry case

$$\mathbb{B}_{a,b}(\bar{u};\bar{v}) =$$
$$= \sum g(\bar{v}_{\rm I},\bar{u}_{\rm I}) \frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II}) g(\bar{v}_{\rm II},\bar{v}_{\rm I}) h(\bar{u}_{\rm I},\bar{u}_{\rm I})}{\lambda_2(\bar{u})\lambda_2(\bar{v}_{\rm II}) f(\bar{v},\bar{u})} \, \mathbb{T}_{13}(\bar{u}_{\rm I})\, T_{12}(\bar{u}_{\rm II})\, \mathbb{T}_{23}(\bar{v}_{\rm II})\Omega.$$

The sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\bar{v}_{\rm II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_{\rm I},\bar{u}_{\rm II}\}$ with the restriction $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = n$, where $n = 0, 1, \dots, \min(a, b)$. Ω is a pseudovacuum and λ_j are eigenvalues of T_{ii} such that

$$T_{jj}(z)\Omega = \lambda_j(z)\Omega; \qquad T_{ij}\Omega = 0, \qquad i > j.$$

Twisted transfer matrix is

$$\mathcal{T}_\kappa(z) = \sum_{j=1}^{3} (-1)^{[j]} \kappa_j T_{jj}(z),$$

Twisted on-shell Bethe vectors are eigenvectors of the $\mathcal{T}_\kappa(u)$

$$\mathcal{T}_\kappa(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \tau_\kappa(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}),$$

where $\bar{u}$ and $\bar{v}$ satisfy twisted Bethe equations and

$$\tau_\kappa(z) \equiv \tau_\kappa(z|\bar{u},\bar{v}) = \kappa_1\lambda_1(z)f(\bar{u},z) + \kappa_2\lambda_2(z)f(z,\bar{u})f(\bar{v},z) - \kappa_3\lambda_3(z)f(\bar{v},z).$$

If we put all $\kappa_j = 1$, then we will obtain on-shell Bethe vectors.

Bethe equations

Twisted Bethe equations for $\mathfrak{gl}(2|1)$ case

$$r_1(u_j) = \frac{\kappa_2}{\kappa_1}\frac{f(u_j,\bar{u}_j)}{f(\bar{u}_j,u_j)} f(\bar{v},u_j), \qquad j = 1,\dots,a,$$

$$r_3(v_k) = \frac{\kappa_2}{\kappa_3} f(v_k,\bar{u}), \qquad k = 1,\dots,b.$$

where

$$r_1(u) = \lambda_1(u)/\lambda_2(u), \qquad r_3(v) = \lambda_3(v)/\lambda_2(v).$$

κ_i — parameters of twist.

Scalar product

Introduce a dual Bethe vector

$$\mathbb{C}_{a,b}(\bar{u}^C; \bar{v}^C) = \mathbb{B}_{a,b}(\bar{u}^C; \bar{v}^C)^\dagger$$

Scalar product of a Bethe vector and a dual Bethe vector is defined as follows:

$$S_{a,b} \equiv S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B) = \mathbb{C}_{a,b}(\bar{u}^C; \bar{v}^C)\mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B).$$

Determinant formula. Explicit form

If $\{\bar{u}^C, \bar{v}^C\}$ satisfy the twisted Bethe equations and $\{\bar{u}^B, \bar{v}^B\}$ satisfy the Bethe equations, then we obtain

$$S_{a,b} = H(\bar{u}^B, \bar{u}^C, \bar{v}^C) \det_{a+b} \begin{pmatrix} \mathcal{N}^{(11)} & \mathcal{N}^{(12)} \\ \mathcal{N}^{(21)} & \mathcal{N}^{(22)} \end{pmatrix},$$

where

$$H(\bar{u}^B, \bar{u}^C, \bar{v}^C) = (-1)^{\frac{b^2+b}{2}} \Delta_{a+b}(\{\bar{u}^B, \bar{v}^C\}) \Delta'_a(\bar{u}^C) \Delta'_b(\bar{v}^C).$$

$$\mathcal{N}_{jk}^{(11)} = (-1)^{a-1}\frac{r_1(u_k^B)}{f(\bar{v}^C, u_k^B)} t(u_j^C, u_k^B) h(\bar{u}^C, u_k^B) + \frac{\kappa_2}{\kappa_1} t(u_k^B, u_j^C) h(u_k^B, \bar{u}^C),$$

$$\mathcal{N}_{jk}^{(12)} = \frac{\kappa_2}{\kappa_1} t(v_k^C, u_j^C) h(v_k^C, \bar{u}^C),$$

$$\mathcal{N}_{jk}^{(21)} = h(u_k^B, \bar{u}^B)\frac{g(\bar{v}^B, u_k^B)}{g(\bar{v}^C, u_k^B)}\left(g(u_k^B, v_j^C) + \frac{\kappa_1}{\kappa_3}\frac{1}{h(v_j^C, u_k^B)}\right),$$

$$\mathcal{N}_{jk}^{(22)} = \delta_{jk}\left(1 - \frac{\kappa_2}{\kappa_3}\frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)}\right) h(v_k^C, \bar{u}^B)\frac{g(v_k^C, \bar{v}^B)}{g(v_k^C, \bar{v}_k^C)}.$$

Form factors of the monodromy matrix elements are defined as

$$\mathcal{F}^{ij}(z) = \mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C)\, T_{ij}(z) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B).$$

Determinant representaion for "on-shell"-"twisted on-shell" scalar product allows us to calculate determinant formulas for form factors.