KPZ growth equation and directed polymers universality and integrability

P. Le Doussal (LPTENS)

with : Pasquale Calabrese (Univ. Pise, SISSA)

Alberto Rosso (LPTMS Orsay)

Thomas Gueudre (LPTENS,Torino)

Andrea de Luca (LPTENS,Orsay)

- growth processes, FPP, Eden, DLA: (tuesday, in random geometry QLE)
- in plane, local rules -> 1D Kardar-Parisi-Zhang class (integrability)

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- in plane, local rules -> 1D Kardar-Parisi-Zhang class (integrability)

- many discrete models in “KPZ class” exhibit universality related to random matrix theory: Tracy Widom distributions: of largest eigenvalue of GUE,GOE..

=> solution continuum KPZ equation (at all times)
+ equivalent directed polymer problem

Replica Bethe Ansatz method:
integrable systems (Bethe Ansatz) +disordered systems(replica)

in math: discrete models => allowed rigorous replica

Part I : KPZ/DP: Replica Bethe Ansatz (RBA)

- KPZ equation, KPZ class, random matrices,Tracy Widom distributions.
- solving KPZ at any time by mapping to directed paths
  then using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => large time TW distrib. for KPZ height

- droplet initial condition => GUE
- flat initial condition => GOE
- half space initial condition => GSE
- stationary (Brownian) initial condition => Baik-Rains

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Part II: N non-crossing directed polymers

Generalized Bethe-ansatz

=> N largest eigenvalues GUE

Macdonald process (Borodin-Corwin)

Andrea de Luca, PLD, arXiv1606.08509,
Phys. Rev. E 93, 032118 (2016) and 92, 040102 (2015)

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2}$ $x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

even at large time PDF depends on some details of initial condition

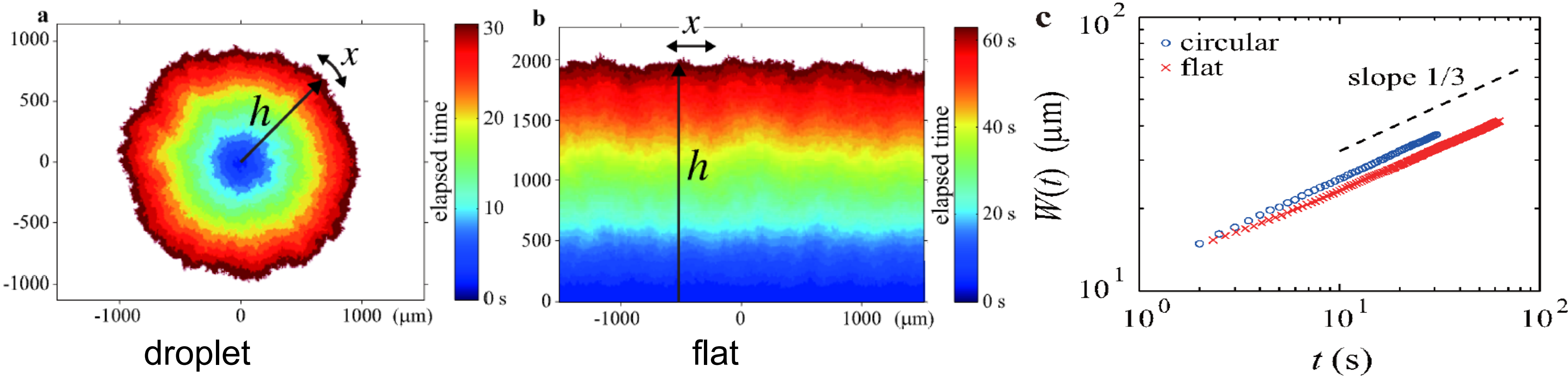
related to RMT

flat	h(x,0) =0
wedge (droplet)	h(x,0) = - w |x|

$\lambda_0 = 0$ Edwards Wilkinson P(h) gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t \to +\infty} v_\infty t + \chi \, t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

also reported in:

- slow combustion of paper — J. Maunuksela et al. PRL 79 1515 (1997)
- bacterial colony growth — Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)
- fronts of chemical reactions — S. Atis (2012)
- formation of coffee rings via evaporation — Yunker et al. PRL (2012)

Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $i = 1, ..N$

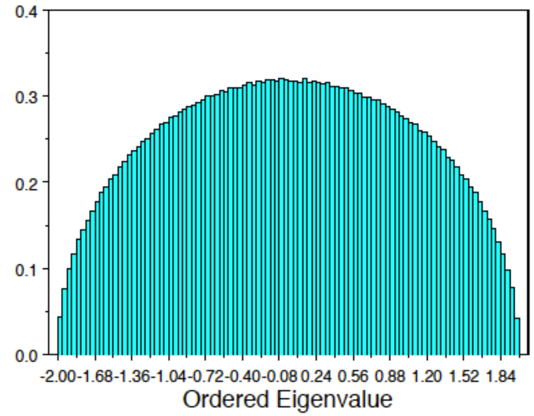
$$P[\lambda] = c_{N,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

H is:

$\beta =$
- 1 (GOE) real symmetric
- 2 (GUE) hermitian
- 4 (GSE) symplectic

Universality large N :

- DOS: semi-circle law



histogram of eigenvalues N=25000

- distribution of the largest eigenvalue

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

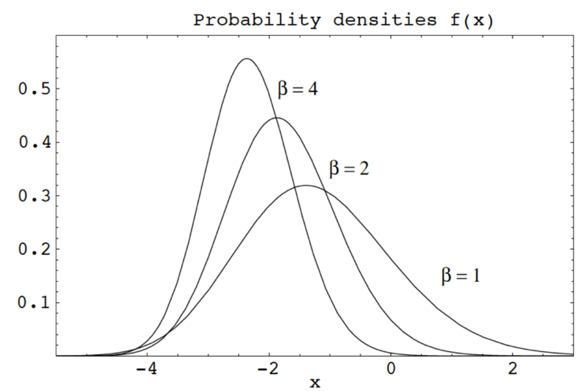
Tracy-Widom distributions (largest eigenvalue of RM)

GOE $$F_1(s) = Det[I - K_1]$$

Fredholm determinants

$$K_1(x,y) = \theta(x) Ai(x+y+s)\theta(y)$$

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$



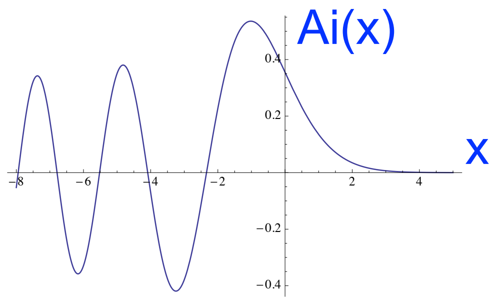
GUE $$F_2(s) = Det[I - K_2]$$

$$K_2(x,y) = K_{Ai}(x+s, y+s)$$

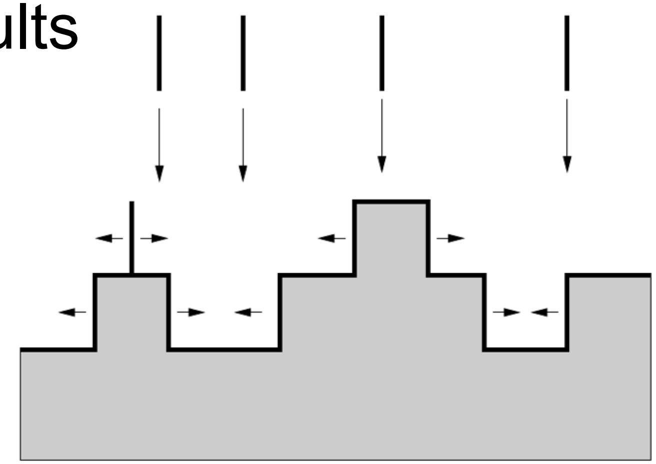
$$K_{Ai}(x,y) = \int_{v>0} Ai(x+v)Ai(y+v)$$

Ai(x-E)

is eigenfunction E
particle linear potential



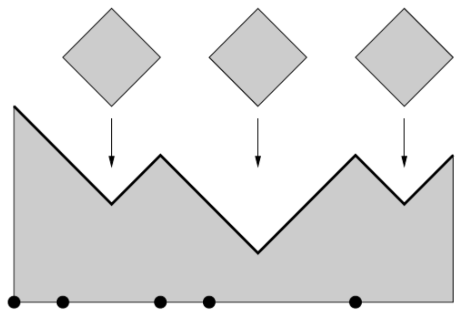
discrete models in KPZ class/exact results

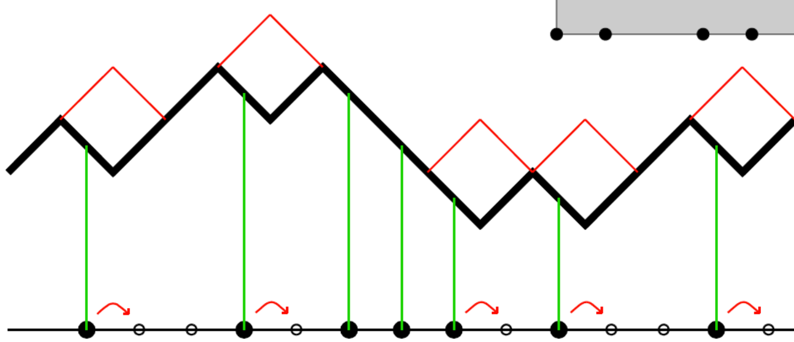


- polynuclear growth model (PNG)

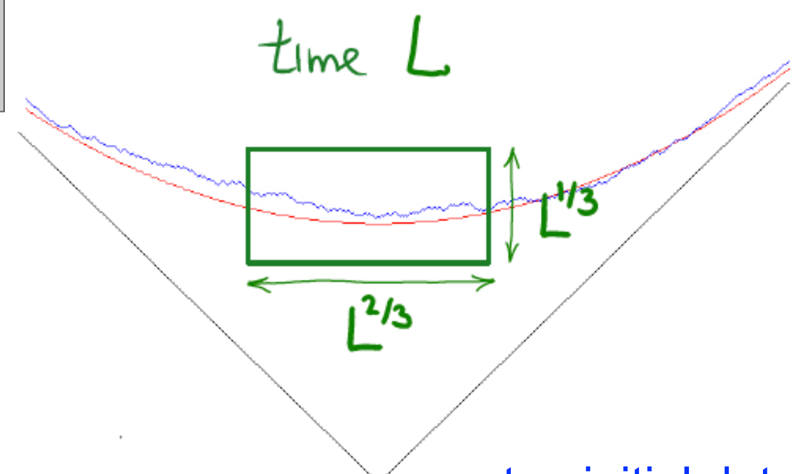
  Prahofer, Spohn, Baik, Rains (2000)

- totally asymmetric exclusion process (TASEP)





Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as $\exp(-x)dx$.



step initial data

Johansson (1999)

Exact results for height distributions for some discrete models in KPZ class

- PNG model

Baik, Deft, Johansson (1999)

$$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$$

droplet IC
GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$A_1(y)$ GOE

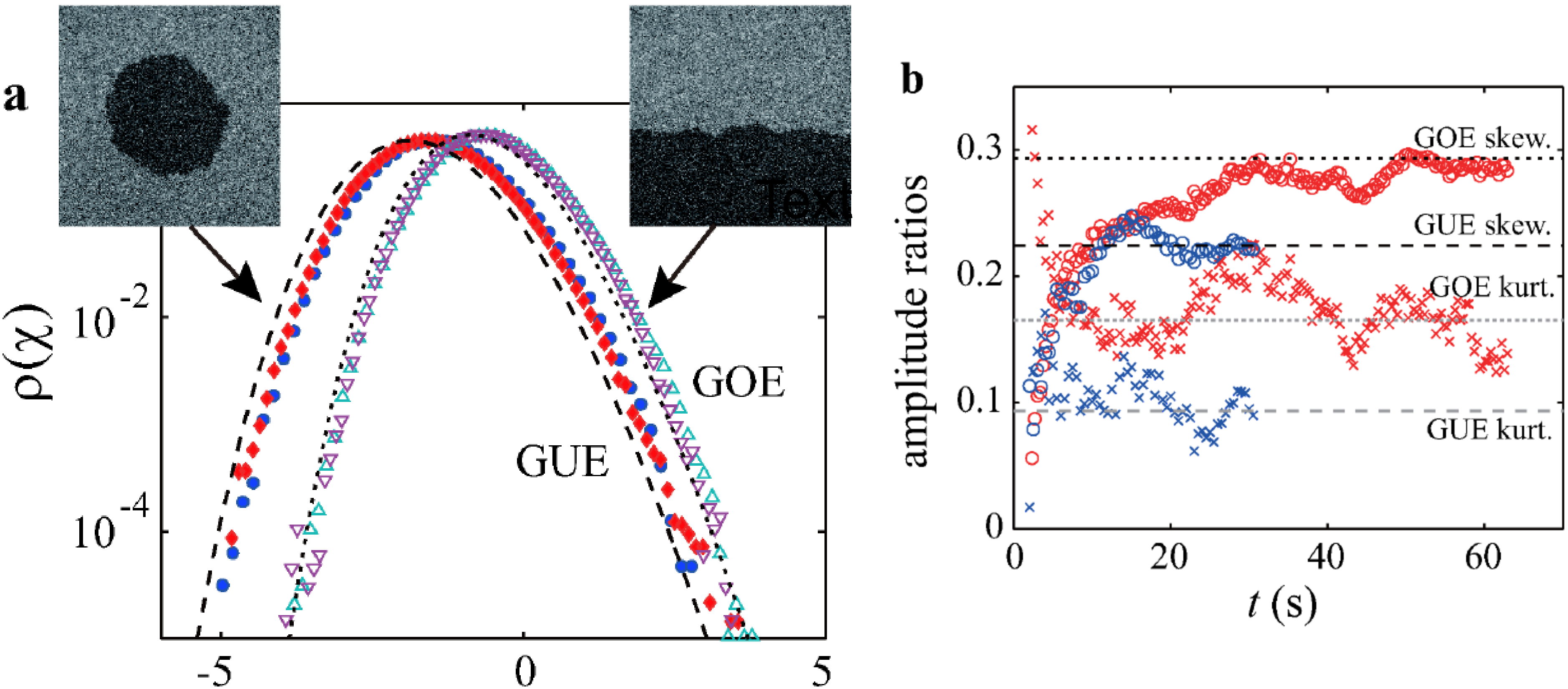
$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3}A_n(y)$$

- similar results for TASEP

Johansson (1999), ...

$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi,$$

skewness = $$\frac{<(h-<h>)^3>}{<(h-<h>)^2>^{3/2}}$$



solving KPZ equation: is KPZ equation in KPZ class ?

Cole Hopf mapping

KPZ equation

Continuum
Directed paths (polymers)
in a random potential

Quantum mechanics
of bosons
(imaginary time)

Kardar 87

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
  Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
  Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

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- **Flat** KPZ/Continuum DP one free endpoint (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Ortmann, J. Quastel and D. Remenik arXiv1407.8484 and arXiv 1503.05626

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- **Stationary** KPZ

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x,t)$$

define:

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

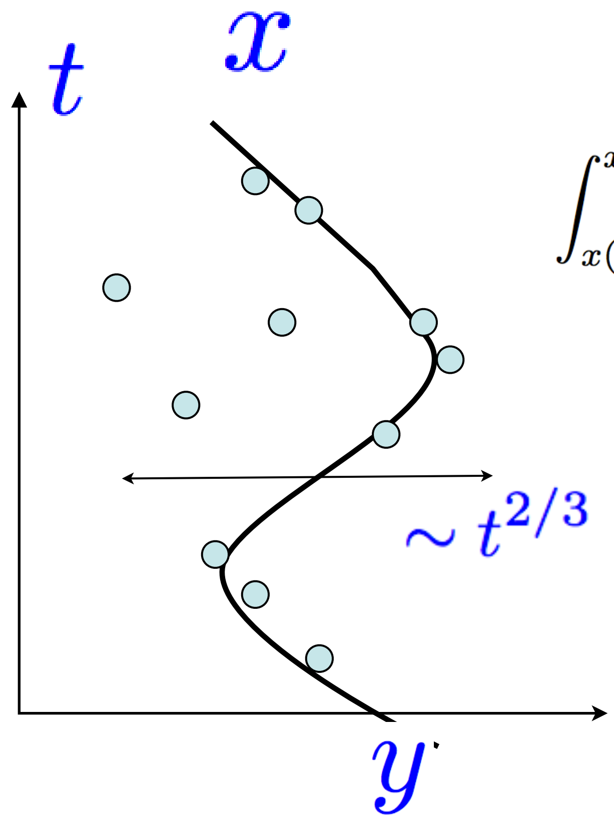
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2} \partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)



$$Z(x,t|y,0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau\, \frac{\kappa}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau),\tau)}$$

$$\overline{V(x,t)V(x',t')} = \bar{c}\;\; \delta(t-t')\delta(x-x')$$

Feynman Kac

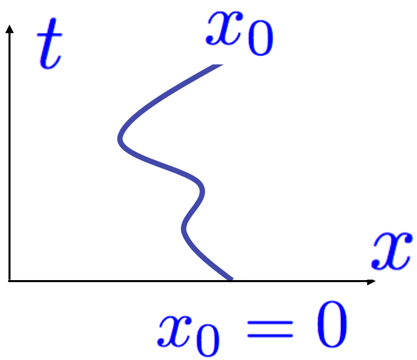
$$Z(x,y,t=0) = \delta(x-y)$$

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

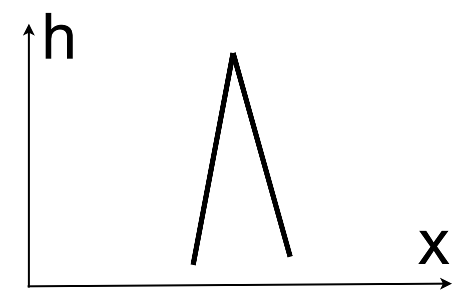
1) DP both fixed endpoints $Z(x_0, t|x_0, 0)$



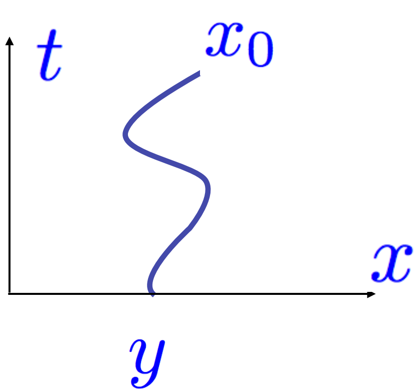
KPZ: narrow wedge <=> droplet initial condition

$$h(x, t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0, t|y, 0)$



KPZ: flat initial condition

$$h(x, t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

calculate $$\overline{Z^n} = \int dZ\, Z^n P(Z) \qquad n \in \mathbb{N}$$

“guess” the probability distribution from its integer moments:

$$P(Z) \to P(\ln Z) \to P(h)$$

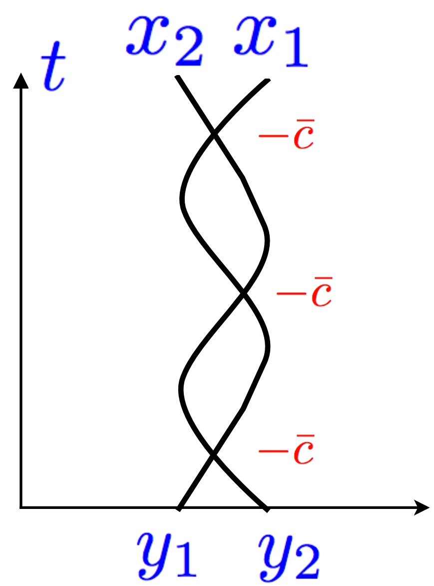
Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1,t|y_1,0)..Z(x_n,t|y_n0)} = \langle x_1,..x_n|e^{-tH_n}|y_1,..y_n\rangle$$

$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

drop the tilde..



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\le i<j\le n} \delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963) what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0 ... x_0|e^{-tH_n}|x_0, ..x_0 >$$

symmetric states = bosons

$$= \sum_\mu \Psi^*_\mu(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

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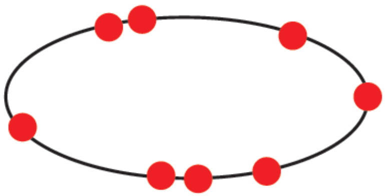
symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2} e^{-E_\mu t}$$

- flat initial condition

$$\overline{(\int_y Z(x_0 t|y0))^n} = \sum_\mu \Psi_\mu^*(x_0, .x_0) \int_{y_1,.y_n} \Psi_\mu(y_1, .y_n) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

LL model: n bosons on a ring with local delta attraction



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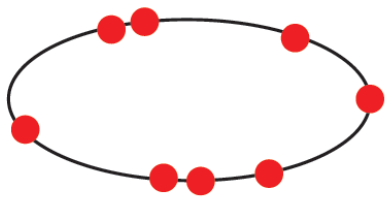

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Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2$$

$$A_P = \prod_{n \ge \ell > k \ge 1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

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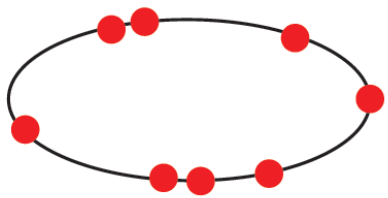

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They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell \ne j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1, ..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$ exponent 1/3

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exponent 1/3

can it be continued in n ? NO !

information about the tail
of the distribution of “free energy” f = - ln Z = - h

$$P(f) \sim_{f\to-\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

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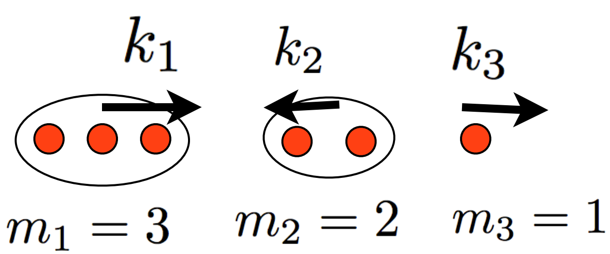
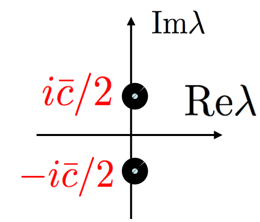
$$E_0(n) = -\frac{\bar{c}^2}{12} n(n^2-1)$$

need to sum over all eigenstates !

- all eigenstates are: All possible partitions of n into ns “strings” each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$\lambda_{j,a_j} = k_j + \frac{i\bar{c}}{2}(m_j + 1 - 2a_j) \qquad a_j = 1, ..m_j$$
$$j = 1, ..n_s$$

=> $$E_\mu = \sum_{j=1}^{n_s} (m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_\mu \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

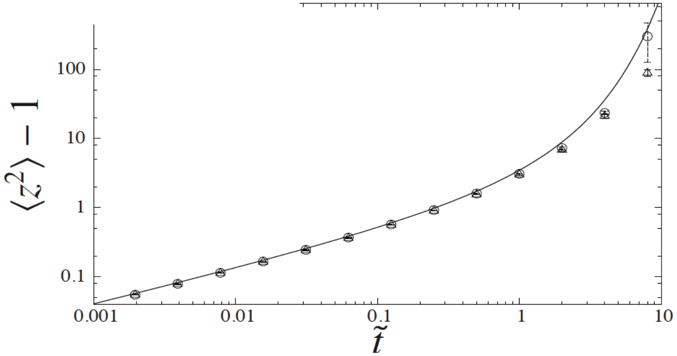
$$\Psi_\mu(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\dots m_{n_s})_n} \qquad n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2 c^2/4}{(k_i-k_j)^2 + (m_i+m_j)^2 c^2/4}$$



how to get P(ln Z) i.e. P(h) ?

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\ln Z = -\lambda f$$

f = - ln Z = - h random variable expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f - x)} = Prob(f > x)$$

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f = - ln Z = h

random variable
expected O(1)

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what we aim
to calculate=
Laplace transform
of P(Z)

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f - x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\le i<j\le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

reorganize sum over number of strings

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Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\le i<j\le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

double Cauchy formula

$$det[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)}]$$

$$= \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s, x) = \prod_{j=1}^{n_s} \int_{v_j>0} dv_j \; det[K(v_j, v_\ell)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1, v_2) = \; - \int \frac{dk}{2\pi} dy Ai(y + k^2 - x + v_1 + v_2) e^{-ik(v_1 - v_2)} \; \frac{e^{\lambda y}}{1 + e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x) \; = Det[I + K]$$

by an equivalent definition
of a Fredholm determinant

$$K(v_1, v_2) \equiv \theta(v_1) K(v_1, v_2) \theta(v_2)$$

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2) large time limit $\lambda = +\infty$ $\quad \frac{e^{\lambda y}}{1+e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2+v+v') e^{ik(v-v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

g(x)= $Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v,v') \; = \int_{y>0} Ai(v+y) Ai(v'+y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i+k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i - m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i+m_i-2ik_j-m_j}{2ik_i+m_i+2ik_j+m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j>0} \mathrm{Pf}[\mathbf{K}(v_i,v_j)]_{2n_s,2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x+y+s)\theta(y)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) [\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$+ \frac{\pi \delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2} \int_y Ai(y + s + v_i)(e^{-2e^{\lambda y}} - 1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i - v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z \, {}_1F_2 \left(1; 2 - 2ik, 2 + 2ik; -z\right)}{\sinh\left(2\pi k\right) \Gamma\left(2 - 2ik\right) \Gamma\left(2 + 2ik\right)}\,, \qquad (19)$$

$$F(z_i, z_j) = \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2 (1-u)})[z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)]\,.$$

$$g_\lambda(s) = \sqrt{Det(1 - 2K_{10})}(1 + \langle \tilde{K} | (1 - 2K_{10})^{-1} | \delta \rangle)$$

$$K_{10}(v_1, v_2) = \partial_{v_1} K_{11}(v_1, v_2) \qquad K_{12}(v_1, v_2) = \tilde{K}(v_1)\delta(v_2)$$

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large time limit

$$\lim_{\lambda \to +\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda \to +\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) =$$

$$\theta(y_1 + y_2)(\theta(y_1)\theta(-y_2) - \theta(y_2)\theta(-y_1))$$

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Summary: we found

for droplet initial conditions $$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions $$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + (\frac{t}{t^*})^{1/3}\chi$$

similar (more involved)

χ at large time has the same distribution as the largest eigenvalue of the GOE

$$t^* = \frac{8(2\nu)^5}{D^2\lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

large for weak noise, large diffusivity

GSE ?

Summary:

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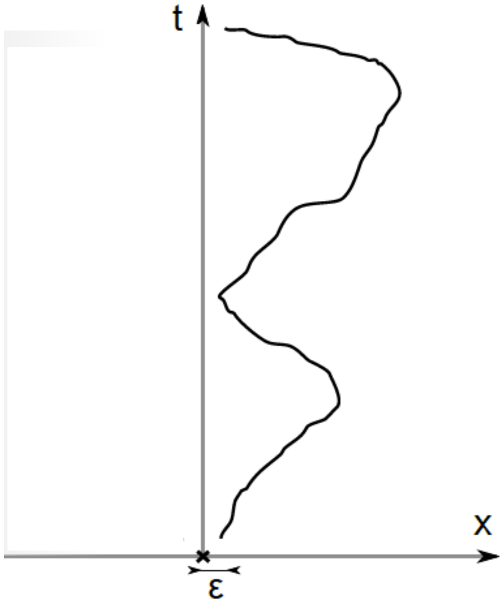
GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal,
EPL 100 26006 (2012)



$$g(s) = \sqrt{\mathrm{Det}[I + \mathcal{K}]}$$

$$\mathcal{K}(v_1, v_2) = -2\theta(v_1)\theta(v_2)\partial_{v_1} f(v_1, v_2)$$

$$f(v_1, v_2) = \int \frac{dk}{2\pi} \int_y Ai(y + s + v_1 + v_2 + 4k^2)\, f_{k/\lambda}(e^{\lambda y}) \frac{e^{-2ik(v_1 - v_2)}}{2ik}$$

$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right)$$
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

$\nabla h(0, t)$ fixed

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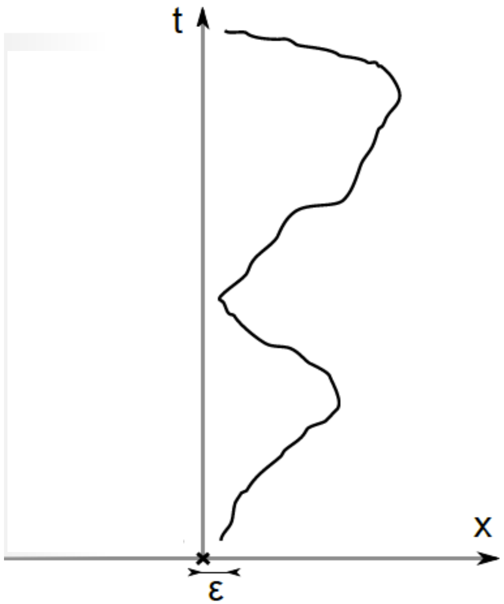

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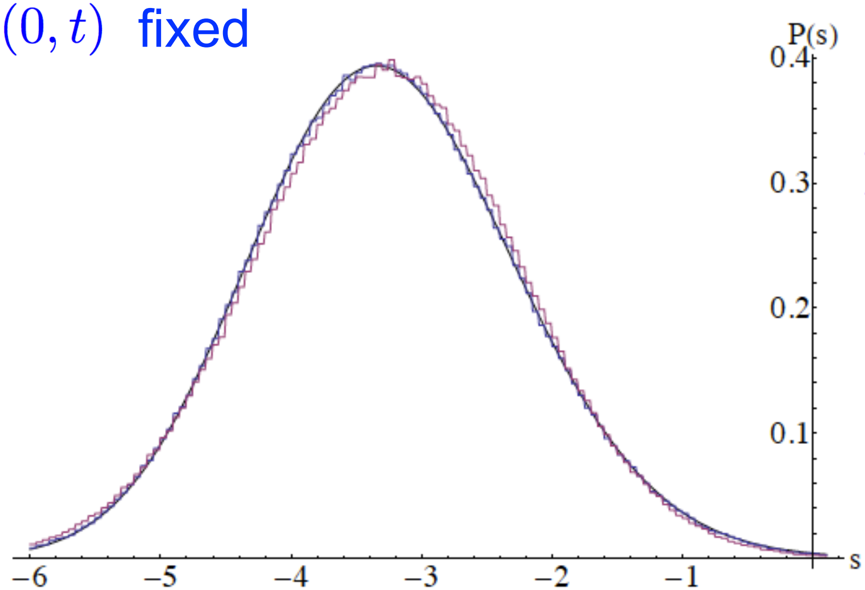
$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$

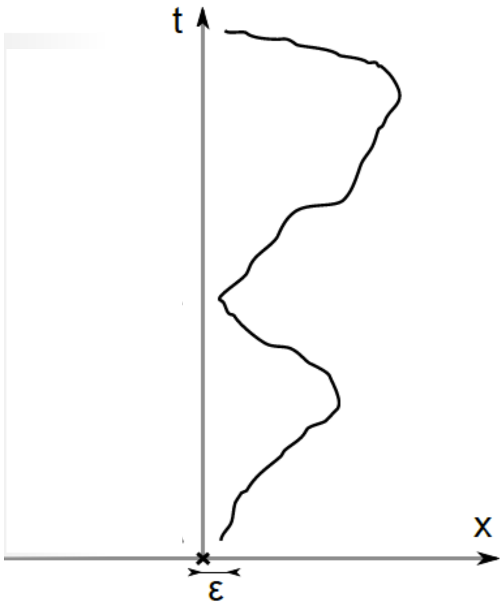


$$\ln Z = \frac{\lambda_0}{2\nu}\tilde{h}(0, t) = v_\infty t + 2^{2/3}\lambda\chi_4$$

χ_4 distributed as $F_4(s)$

Gaussian Symplectic Ensemble

Probability that a polymer (starting near the wall) does not cross the wall



$$q_\eta(t) = \frac{Z_\eta^{\text{half space}}(t)/\epsilon^2}{Z_\eta^{\text{full space}}(t)}$$

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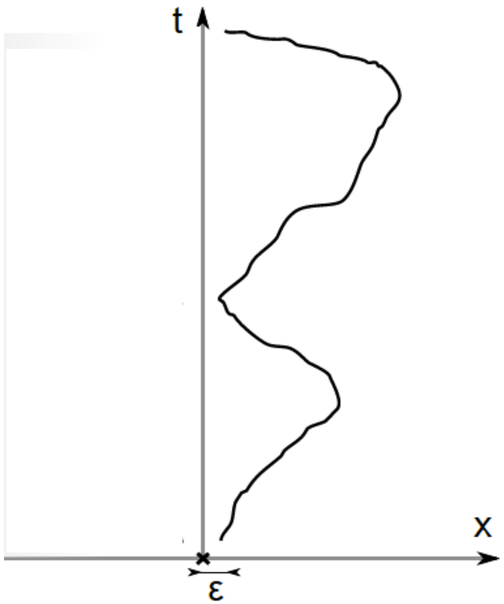

$$q_\eta(t) = \frac{Z_\eta^{\text{half space}}(t)/\epsilon^2}{Z_\eta^{\text{full space}}(t)}$$

$$\overline{\ln q_\eta(t)} = -(\overline{\chi_2} - \overline{\chi_4})(\bar{c}^2 t)^{1/3} \approx -1.49134(\bar{c}^2 t)^{1/3}$$

$$\mu^{F_2} = -1.7710868 \qquad \mu^{F_4} = -3.2624279$$

gives q(t) in typical sample: decays sub-exponentially

Part II: non-crossing directed polymers

with Andrea de Luca (LPTENS,Orsay, Oxford)

Conjecture about N mutually avoiding paths in random potential

$\hat{\mathcal{Z}}_1(t)$ continuum partition sum of one directed polymer w. fixed endpoints at 0

$$\ln \hat{\mathcal{Z}}_1(t) \simeq -t/12 + \hat{\gamma}_1 t^{-1/3}$$

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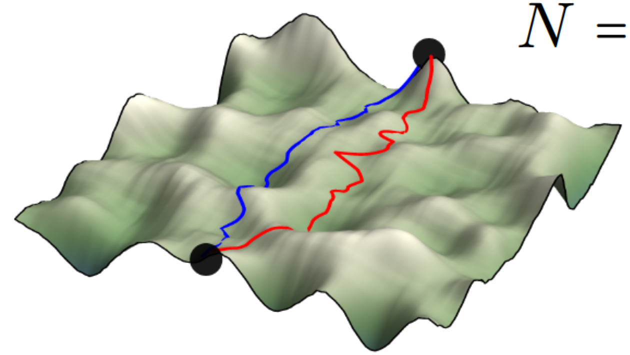

$N = 2$

Conjecture about N mutually avoiding paths in random potential

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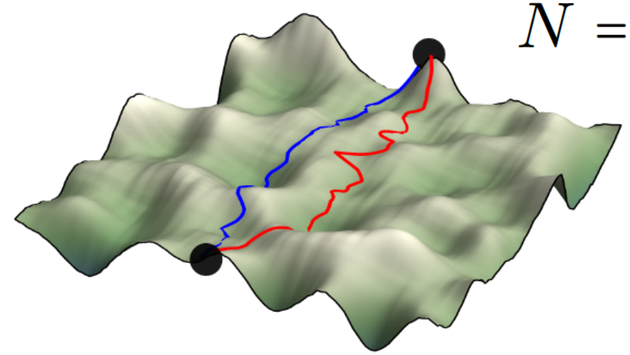
CONJECTURE:

$$\ln \hat{\mathcal{Z}}_N(t) \simeq -Nt/12 + t^{1/3} \hat{\zeta}^{(N)}$$



$$\hat{\zeta}^{(N)} \overset{\text{in law}}{\equiv} \sum_{i=1}^{N} \hat{\gamma}_i =: \hat{\gamma}$$

$\hat{\gamma}_1, \ldots, \hat{\gamma}_N$ N largest eigenvalues of a GUE random matrice

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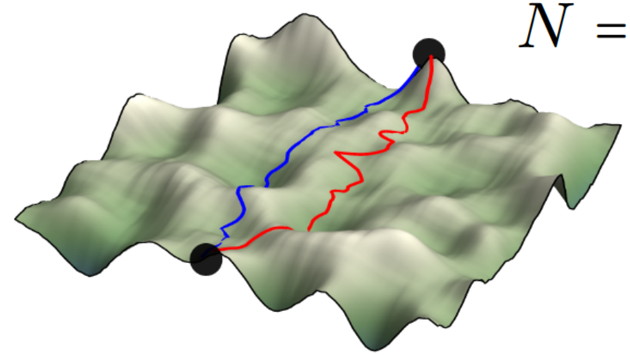

$\hat{\gamma}_1, \ldots, \hat{\gamma}_N$ N largest eigenvalues of a GUE random matrice

T=0 semidiscrete DP model Yor, O' Connell, Doumerc (2002)

Warren, O' Connell, Lun (2015) Corwin, Nica (2016)

CONJECTURE: $$\ln \hat{Z}_N(t) \simeq -Nt/12 + t^{1/3} \hat{\zeta}^{(N)}$$

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We show by explicit calculation for large positive argument

$$P_N^{\mathrm{DP}}(\zeta) = \rho_N^{\mathrm{DP}}(\zeta)(1 + O(e^{-a_N \zeta^{3/2}}))$$

$$P_N^{\mathrm{GUE}}(\gamma) = \rho_N^{\mathrm{GUE}}(\gamma)(1 + O(e^{-a'_N \gamma^{3/2}}))$$

$$\rho_N^{\mathrm{DP}}(\gamma) = \rho_N^{\mathrm{GUE}}(\gamma)$$

The tail approximants exactly match

$$= O(e^{-\frac{4\gamma^{3/2}}{3\sqrt{N}}})$$

PDF of sum of GUE largest eigenvalues

GUE random matrix $\mathcal{N} \times \mathcal{N}$ eigenvalues $\hat{\lambda}_1 > \ldots > \hat{\lambda}_{\mathcal{N}} \longrightarrow (-\sqrt{2\mathcal{N}}, \sqrt{2\mathcal{N}})$

scaled eigenvalues near the edge $\hat{\gamma}_l = (\hat{\lambda}_l - \sqrt{2\mathcal{N}})\sqrt{2}\mathcal{N}^{1/6}$

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JPDF of N largest $\boldsymbol{\gamma}_N$ stands for $\gamma_1, \ldots, \gamma_N$

$$p_N(\boldsymbol{\gamma}_N) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{j=1}^{k} \int_{\min_{i=1}^{N} \gamma_i}^{\infty} dx_j \, r_{N+k}(\boldsymbol{\gamma}_N, \mathbf{x}_k)$$

N point correlation $r_N(x_1, \ldots, x_N) = \det[K_{\mathrm{Ai}}(x_i, x_j)]_{i,j=1}^{N}$

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tail approximant

$$\simeq r_N(\boldsymbol{\gamma}_N)$$

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$$\sum_{i=1}^{N} \hat{\gamma}_i =: \hat{\gamma} \qquad \overline{\exp(u\hat{\gamma})} = \frac{1}{N!} \prod_{i=1}^{N} \int_0^{\infty} dv_i \det\Big[\int_{-\infty}^{\infty} d\gamma e^{\gamma u} \, \mathrm{Ai}(\gamma + v_j) \, \mathrm{Ai}(\gamma + v_k)\Big]_{j,k=1}^{N}$$

PDF of sum of GUE largest eigenvalues

GUE random matrix $\mathcal{N} \times \mathcal{N}$ eigenvalues $\hat{\lambda}_1 > \ldots > \hat{\lambda}_{\mathcal{N}} \longrightarrow (-\sqrt{2\mathcal{N}}, \sqrt{2\mathcal{N}})$

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tail approximant

$$p_N(\boldsymbol{\gamma}_N) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{j=1}^{k} \int_{\min_{i=1}^N \gamma_i}^{\infty} dx_j \, r_{N+k}(\boldsymbol{\gamma}_N, \mathbf{x}_k) \qquad \simeq r_N(\boldsymbol{\gamma}_N)$$

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$$P_N^{\mathrm{GUE}}(\gamma) = \rho_N^{\mathrm{GUE}}(\gamma)(1 + O(e^{-a'_N \gamma^{3/2}}))$$

$$\tilde{\rho}_N^{\mathrm{GUE}}(u) = \frac{e^{\frac{Nu^3}{12}} u^{-\frac{3N}{2}}}{\pi^{N/2} N!} \prod_{i=1}^{N} \int_{v_i > 0} e^{-2v_i} \det\left[e^{-\frac{(v_j - v_k)^2}{u^3}}\right]_{j,k=1}^N$$

Laplace transform of tail approximant

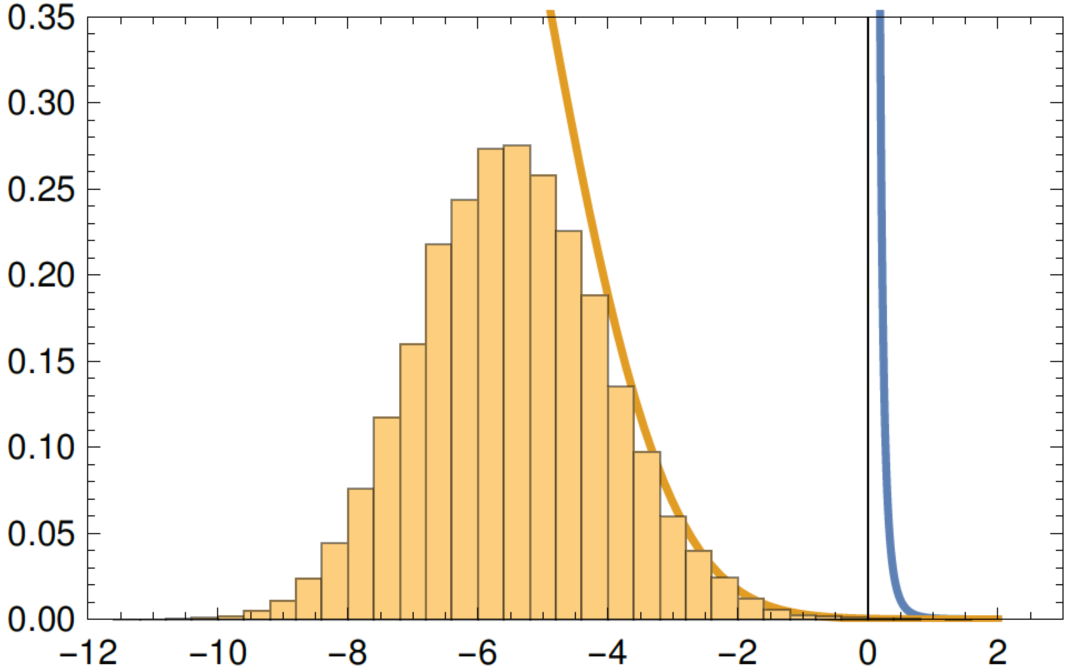

FIG. 2. (Color online) The empirical distribution for the sum of the first $N = 2$ eigenvalues in 10^5 realizations of GUE matrices of size $\mathcal{N} = 250$. The continuous line are two different approximations for the tail obtained: from the inverse Laplace transformation of Eq.(12) (orange); from the simple approximation $R_N(\gamma) = 1$ in Eq. (16) (blue).

N non-crossing directed paths in a random potential

Partition sum of 1 path
with endpoints y,x

$$Z_\eta(x;y|t)$$

Karlin McGregor formula

Partition sum of N non-crossing paths
with endpoints

$y_1 < y_2 < ..y_N \qquad x_1 < x_2 < ..x_N$

$$\det[Z_\eta(x_i;y_j|t)]_{N\times N}$$

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limit of coinciding endpoints $\qquad \epsilon \to 0$

$$x_i = y_i = \epsilon u_i \qquad \hat{Z}_\eta^{(N)}(\mathbf{x};\mathbf{y}|t) \simeq \frac{\epsilon^{N(N-1)}}{G(N+1)^2}\prod_{i<j}(u_i-u_j)^2 \hat{\mathcal{Z}}_N(t)$$

$$\hat{\mathcal{Z}}_N(t) = \det[\partial_x^{i-1}\partial_y^{j-1}\hat{Z}_\eta(x;y|t)|_{x=y=0}]_{i,j=1}^N$$

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Warren, O' Connell arXiv1104.3509

Warren, Lun (2015) Corwin, Nica (2016)

Final formula for m-th moment

can be expressed as a sum over eigenstates of Lieb-Liniger model (strings)

$$\overline{\hat{\mathcal{Z}}_N(t)^m} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi)^{n_s}} \sum_{(m_1,\dots m_{n_s})_n}$$

$$\prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j}{m_j} e^{-tE[\mathbf{k},\mathbf{m}]} \Phi[\mathbf{k},\mathbf{m}] \mathcal{B}_{N,m}[\mathbf{k},\mathbf{m}]$$

$n = mN$ particles

(replica..)

n_s strings of sizes $m_1, \dots, m_{n_s}$

$\sum_{j=1}^{n_s} m_j = n \qquad m_j \geq 1$

$$E[\mathbf{k},\mathbf{m}] = \sum_{j=1}^{n_s} m_j k_j^2 + \frac{1}{12}(m_j - m_j^3)$$

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How does one get this formula ?

1) Generalized Bethe Ansatz

2) Residue expansion from a CI formula

Borodin Corwin, Macdonald processes

1) Non-crossing polymers via replica Bethe Ansatz

Andrea de Luca, PLD, arXiv 1505.04802, Phys. Rev. E 92, 040102 (2015)

$N = 2$

$$Z_\eta^{(2)}(\epsilon) = Z_\eta(\epsilon;\epsilon|t)Z_\eta(-\epsilon;-\epsilon|t) - Z_\eta(-\epsilon;\epsilon|t)Z_\eta(\epsilon;-\epsilon|t)$$

$$\Theta_{n,m}(t) \equiv \lim_{\epsilon\to 0} \overline{[(2\epsilon)^{-2} Z_\eta^{(2)}(\epsilon)]^m [Z_\eta(0;0|t)]^{n-2m}}$$

$\longrightarrow$ here n=2 m

n=0 gives moments of non-crossing probability

$$\overline{p_\eta(t)^m} = \lim_{n\to 0} \Theta_{n,m}(t)$$

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quantum mechanics …

Lieb-Liniger model with general symmetry (beyond bosons)

$$H_n = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\le i<j\le n} \delta(x_i - x_j)$$

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$$\mathcal{D}_1 = 2^{-1/2}(\partial_{x_1} - \partial_{x_2})\big|_{\mathbf{x}=0}$$

$$|\Psi_m(\epsilon)\rangle = 2^{-m/2}(\otimes_{j=1}^m |\epsilon,-\epsilon\rangle - |-\epsilon,\epsilon\rangle) \otimes |0\ldots 0\rangle$$

bosonic sector gives vanishing contribution

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more general Bethe ansatz

$$\psi_\mu(\mathbf{x}) = \sum_{P,Q\in\mathcal{S}_n} \vartheta_Q(\mathbf{x}) A_Q^P \exp[i\sum_{j=1}^n x_{Q_j}\mu_{P_j}]$$

A_Q^P inside irreducible representation of S_n

$$x_{Q_1} \le x_{Q_2} \ldots \le x_{Q_n}$$

N=2, 2-row Young diagram $\xi = (n-m, m)$

1) Nested Bethe ansatz

C-N Yang PRL 19,1312 (1967)

auxiliary spin chain

$N = 2$

Bethe equations

$\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$

auxiliary rapidities λ_a $a = 1, ..m$

$$\prod_{\substack{b=1 \\ b\neq a}}^{m} \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^{n} \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2},$$

$$\prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^{m} \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}$$

they implement the symmetry
of the wave-function

solved at large L by strings again !

$$\mu_j^a = k_j + \frac{ic}{2}(m_j + 1 - 2a) + \delta_j^a$$

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→ several roots for auxiliary variables => difficult

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2) From BC formula

$$\Theta_{n,m}(t) = \frac{1}{2^m} \int \frac{dz_1}{2\pi} \cdots \int \frac{dz_n}{2\pi} e^{-t\sum_{k=1}^n z_k^2} \times \left(\prod_{1\leq k<j\leq n} f(z_{kj}) \right) \left(\prod_{q=1}^{m} h(z_{2q-1,2q}) \right)$$

$N = 2$

$$h(u) = u(u - ic) \qquad f(u) \equiv u/(u - ic)$$

Borodin Corwin, arXiv11114408,
Prob. Theor. Rel. Fields 158 225 (2014)

$z_{kj} = z_k - z_j$ imaginary part C_j for z_j satisfying $C_{j+1} > C_j + \bar{c}$

we obtained the residue expansion in form of sums over strings => formula for $\mathcal{B}_{N,m}[\boldsymbol{\mu}]$

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N=3 m=3 n=m N=9

only non zero are

$$\begin{array}{|c|c|c|}\hline \mu_1 & \mu_2 & \mu_3 \\ \hline \mu_4 & \mu_5 & \mu_6 \\ \hline \mu_7 & \mu_8 & \mu_9 \\ \hline \end{array}$$

$$(\mu_1, \mu_2, \mu_3) \to (k_1 - i, k_1, k_1 + i)$$

$$(\mu_4, \mu_5, \mu_6) \to (k_2 - i, k_2, k_2 + i)$$

$$(\mu_7, \mu_9, \mu_9) \to (k_3 - i, k_3, k_3 + i)$$

$+$

$$\begin{array}{|c|c|c|}\hline \mu_1 & \mu_2 & \mu_3 \\ \hline \mu_4 & \mu_5 & \mu_6 \\ \hline \mu_7 & \mu_8 \\ \cline{1-2} \mu_9 \\ \cline{1-1} \end{array}$$

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ground state, lowest E

=> dominate at large t

$$\overline{\hat{Z}_N(t)^m}$$

for fixed m

+

μ_1	μ_2	μ_3
μ_4	μ_5	μ_6
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μ_9		

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BUT not sufficient to obtain the PDF of $\hat{Z}_N(t)$ at large t

$+ \ldots$

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BUT not sufficient to obtain the PDF of $\hat{\mathcal{Z}}_N(t)$ at large t

=> however allows to get the TAIL of the PDF

+ ...

simple formula for ground state

$$\mathcal{B}_{N,m}(\mathbf{k}) = \frac{m!^N}{(mN)!}\prod_{1\le i<j\le N}\prod_{p=1}^{m}[(k_i - k_j)^2 + p^2]$$

$$k_{ij} = k_i - k_j$$

N=1 Tail approximant

GUE-Tracy Widom distribution

$$F_2'(\sigma_1) \sim \frac{e^{-\frac{4}{3}\sigma_1^{3/2}}}{8\pi\sigma_1} \qquad \sigma_1 \gg 1$$

$$F_2(\sigma) = \mathrm{Det}[I - P_\sigma K_{\mathrm{Ai}} P_\sigma]$$

Tail approximant: $F_2(\sigma) = F_2^{(1)}(\sigma) + O(e^{-\frac{8}{3}\sigma^{3/2}}) \qquad \sigma \to +\infty$

neglects terms of order $O(K_{\mathrm{Ai}}^2)$

$$F_2^{(1)}(\sigma) \equiv 1 - \mathrm{Tr}[P_\sigma K_{\mathrm{Ai}}] = 1 - \int_\sigma^{+\infty} dv K_{\mathrm{Ai}}(v,v)$$

$$F_2^{(1)}(\sigma) - 1 = O(e^{-\frac{4}{3}\sigma^{3/2}})$$

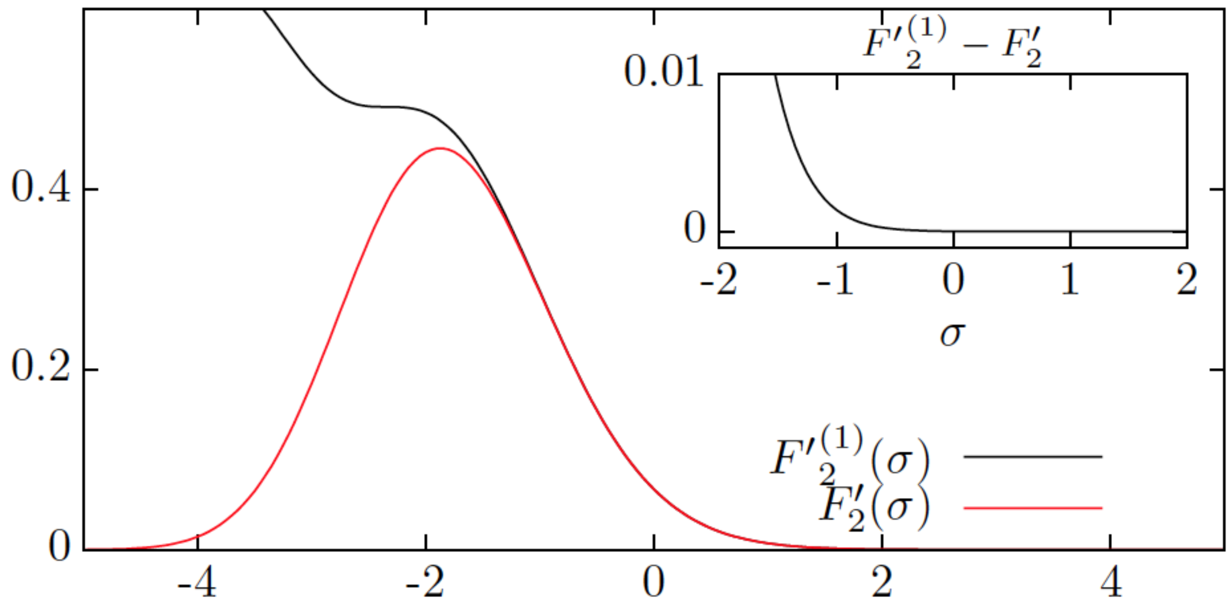

Figure 1. Plot of the Tracy-Widom distribution $F_2'(\sigma)$ (red line) compared with its tail for positive σ given by $F'^{(1)}_2(\sigma)$ (black line). Inset: difference between the Tracy-Widom distribution $F_2'(\sigma)$ and its tail $F'^{(1)}_2(\sigma)$, given in (12).

why is this tail approximant interesting ?

$$F_2^{(1)}(\sigma) \equiv 1 - \mathrm{Tr}[P_\sigma K_{\mathrm{Ai}}] = 1 - \int_\sigma^{+\infty} dv K_{\mathrm{Ai}}(v,v)$$

it corresponds to keeping only contributions of
one n-string when calculating generating function ns=1

<=> n particles all in a single bound state = the ground state
of the Lieb Liniger model

neglects terms of order $O(K_{\mathrm{Ai}}^2)$ contributions of two mj-strings, ..

=> assume this property holds for any N

Tail of the PDF of $\hat{\mathcal{Z}}_N(t)$ at large t

Define a generating function

$$g_N(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m \overline{\hat{\mathcal{Z}}_N(t)^m} = \overline{\exp\left(-e^{-\lambda s + t^{1/3}\hat{\zeta}}\right)}$$

$$x e^{-\frac{Nt}{12}} = e^{-\lambda s}$$

$$\lambda = (Nt/4)^{1/3}$$

$$g_N(s) \underset{t \to \infty}{\to} \mathrm{Prob}(\hat{\zeta} < (N/4)^{1/3} s)$$

keeping only the ground state => tail of the PDF

argument of counting of number of Airy functions

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$$\int_{-\infty}^{\infty} dy\, \mathrm{Ai}(y) e^{yw} = e^{w^3/3}$$

$$g_N^{(0)}(s) \overset{t\to\infty}{=} 1 - \frac{1}{N!} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \frac{dk_i}{2\pi} \int_0^{\infty} dy\, \mathrm{Ai}\Big(y + \sum_i k_i^2 + s\Big)$$
$$\int_{\epsilon - \imath\infty}^{\epsilon + \imath\infty} \frac{dz}{2\pi \imath z} e^{\sqrt{N} z y} \det\left[\frac{1}{2z + \imath k_{jk}}\right]_{j,k=1}^{N} . \quad (27)$$

$$\sum_{m\ge1} (-1)^m \mathsf{f}(m) = -\frac{1}{2i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dz}{\sin(\pi z)} \mathsf{f}(z)$$

$$g_N^{(0)}(s) \overset{t\to\infty}{=} 1 - \frac{1}{N!}\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\frac{dk_i}{2\pi}\int_0^{\infty} dy\,\mathrm{Ai}\Big(y + \sum_i k_i^2 + s\Big)$$

$$\int_{\epsilon-\imath\infty}^{\epsilon+\imath\infty}\frac{dz}{2\pi\imath z}e^{\sqrt{N}zy}\det\left[\frac{1}{2z+\imath k_{jk}}\right]_{j,k=1}^{N} . \quad (27)$$

$$\int_{-\infty}^{\infty} ds\,\mathrm{Ai}'(y + \sum_i k_i^2 + s)e^{\tilde{u}s} = -\tilde{u}e^{-\tilde{u}(\sum_i k_i^2 + y)}e^{\frac{\tilde{u}^3}{3}}$$

$$\tilde{\rho}_N^{\mathrm{DP}}(u) = \int_{-\infty}^{\infty} ds\,\partial_s g_N^{(0)}(s)e^{(\frac{N}{4})^{1/3}us}$$

$$= \frac{e^{\frac{\tilde{u}^3}{3}}}{N!}\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\frac{dk_i}{2\pi}e^{-\frac{\tilde{u}k_i^2}{N}}\det\left[\frac{1}{2\tilde{u}+\imath k_{jk}}\right]_{j,k=1}^{N}$$

$$= \frac{e^{\frac{Nu^3}{12}}u^{-\frac{3N}{2}}}{\pi^{N/2}N!}\prod_{i=1}^{N}\int_{v_i>0} e^{-2v_i}\det\left[e^{-\frac{(v_j-v_k)^2}{u^3}}\right]_{j,k=1}^{N}$$

$$= \tilde{\rho}_N^{\mathrm{GUE}}(u)$$

Conclusion

- showed conjecture that the free energy of N non-crossing paths in continuum converges in law to sum of N GUE largest eigenvalues holds in the tail

Still open

- go beyond the tail

- larger conjecture that JPDF of

$$\frac{1}{t^{1/3}}\left\{\ln\frac{Z_1}{Z_0},\dots,\ln\frac{Z_N}{Z_{N-1}}\right\} \overset{\text{in law}}{\equiv} \{\gamma_1,\dots,\gamma_N\}$$

Perspectives/other works

- replica BA method

Airy process

stationary KPZ Sasamoto Inamura $t \to \infty$ $A_2(y)$

2 space points $Prob(h(x_1,t),h(x_2,t))$ Prohlac-Spohn (2011), Dotsenko (2013)

2 times $Prob(h(0,t),h(0,t'))$ Dotsenko (2013)

endpoint distribution of DP Dotsenko (2012) Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

q-TASEP $q \to 1$ avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

WASEP Bose gas moments as nested contour integrals

- sine-Gordon FT

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- Lattice directed polymers

- FPP- Eden model on fluctuating geometry

=> QLE(8/3,0)

- FPP- Eden model on Z^2 => KPZ

Is there a KPZ^2 formula ?