Limit shapes of the dimer model

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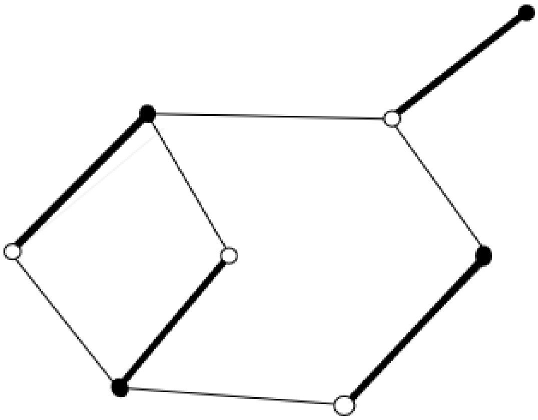
Brief introduction to the Dimer model

Let Γ be a finite planar bipartite graph and $E(\Gamma)$ its set of edges. A *Dimer configuration* on Γ is a subset of edges of Γ such that each vertex of Γ is adjacent to exactly one of these edges. We denote the set of dimer configurations on Γ as $\mathfrak{D}(\Gamma)$.



Gibbs measure on dimer configurations

Boltzmann weight of the edge e,

$$\omega : e \mapsto \omega(e).$$

Boltzmann weight of the dimer configuration D,

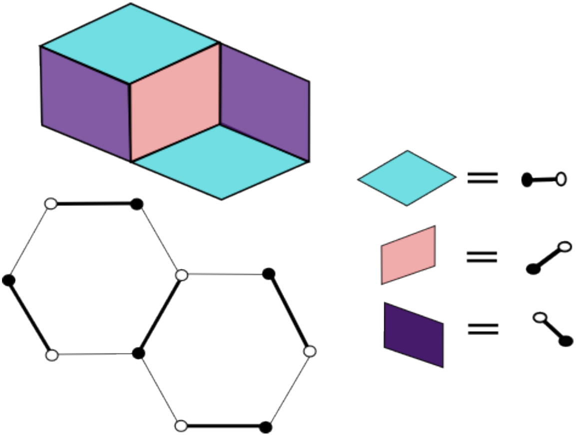
$$W(D) = \prod_{e \in D} \omega(e).$$

Partition function of the dimer model on graph Γ,

$$Z(\Gamma) = \sum_{D \in \mathfrak{D}(\Gamma)} W(D).$$

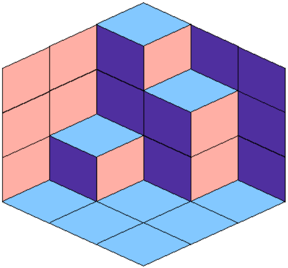
Lozenge tiling and dimers

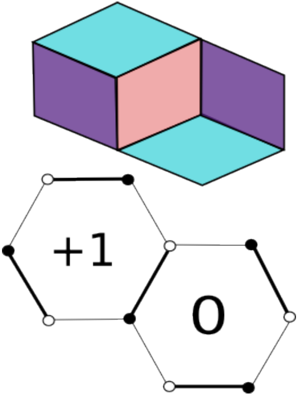
Dimers on the hexagonal lattice dimers are the same as Lozenge tiling.



Height function

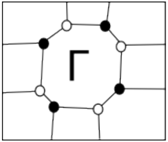
One can assign a function $D \mapsto H_D$ to each dimer configuration that maps every face to its "height". For the hexagonal lattice there is a natural definition.





Height function can be defined for other cases as well.

Periodic graph on torus



Fix fundamental domain Γ.Let T_n be the torus of $n \times n$ fundamental domains,

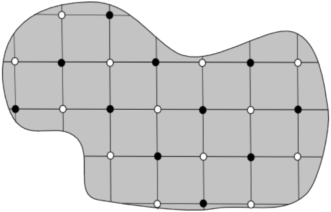
Theorem (Okounkov, Kenyon, Sheffield, 2003)

$$F_n = \lim_{n \to \infty} n^{-2} \log Z(T_n) = \iint_{|z|=|w|=1} \log P(z,w) \frac{dz\, dw}{zw}$$

$P(z,w)$ is a polynomial, which is determined by Γ.
Free energy in magnetic field variables

$$R(X,Y) = \iint_{|z|=1 |w|=1} \log P(ze^X, we^Y) \frac{dz\, dw}{zw}$$

Limit shape for the Dimer model



$\Omega \subset \mathbb{R}^2$, $\Omega_n := \Omega \cap \frac{1}{n}\mathbb{Z}^2$.

Theorem (Cohn, Kenyon, Propp, 2001)

1. $$\lim_{n\to\infty} n^{-2} \log Z(\Omega_n) = \max_h \iint_\Omega \sigma\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) dxdy$$

   $\sigma(s,t)$ is the Legendre transform of $R(X,Y)$, free energy in magnetization variables.

2. *As $n \to \infty$, all height functions converge to g, where g maximizes $\iint_\Omega \sigma\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) dxdy$.*

Theorem (In progress)

Previous theorem is true for any "good" fundamental domain.

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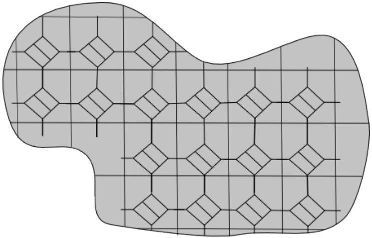
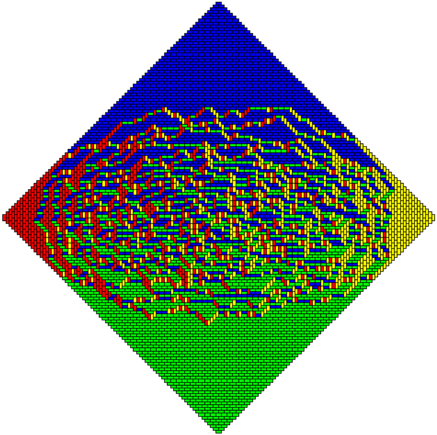


Thank you for your attention!



Merci pour votre attention!