DOUBLE MONOTONE HURWITZ NUMBERS

Random Geometry and Physics
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Definition

Let $H_{g,n}\left(\begin{smallmatrix}\lambda_1,\lambda_2,\dots,\lambda_s\\ \mu_1,\mu_2,\dots,\mu_n\end{smallmatrix}\right)$ be $\frac{1}{|\boldsymbol{\mu}|!}$ multiplied by the number of tuples $(\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_m)$ of elements in $S_{|\boldsymbol{\mu}|} = S_{|\boldsymbol{\lambda}|}$ such that

- σ_0 is of cyclic type $(\lambda_1, \lambda_2, \dots, \lambda_s)$;
- for any $i \in [1, \dots, m]$ the permutation σ_i is a transposition;
- $m = 2g - 2 + n + s$;
- $\sigma_0\sigma_1\sigma_2 \cdots \sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \dots, \mu_n$; and
- $\langle \sigma_0, \sigma_1, \sigma_2, \dots, \sigma_m \rangle$ is transitive.

Double Monotone Hurwitz numbers

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- $\sigma_0\sigma_1\sigma_2\cdots\sigma_m$ has labelled cycles of lengths $\mu_1,\mu_2,\ldots,\mu_n$;
- $\langle\sigma_0,\sigma_1,\sigma_2,\ldots,\sigma_m\rangle$ is transitive; and
- Monotonicity condition: if $\sigma_i=(a_i\ b_i)$ with $a_i<b_i$, then $b_1\leq b_2\leq\cdots\leq b_m$.

Example calculation

Take $(g, n) = (0, 2)$, $\boldsymbol{\lambda} = (1, 1, 1)$ and $\boldsymbol{\mu} = (1, 2)$, so $m = 3$.

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There are 27 products of 3 transpositions in S_3 and 24 are transitive.

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\begin{array}{llll}
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All 24 products produce cycle type $(1, 2)$, so $H_{0,2}\left(\begin{smallmatrix}1,1,1\\1,2\end{smallmatrix}\right) = \frac{24}{3!} = 4$.

Only the first 12 products are monotone, so $H^{\leq}_{0,2}\left(\begin{smallmatrix}1,1,1\\1,2\end{smallmatrix}\right) = \frac{12}{3!} = 2$.

Why to go monotone?

Double Monotone Hurwitz numbers are related to:

- Reperesentation theory (Jucys-Murphy elements);
- Integrable systems;
- Asymptotic expansion of the derivatives of HCIZ-integral

$$\int_{U_N} e^{zA_N U B_N U^{-1}} dU$$

(I.P. Goulden, M. Guay-Paquet, J. Novak *Monotone Hurwitz numbers and the HCIZ integral*' 2014.)

Cut-and-join recursion. Splitting.

Splitting of the numbers of monotone tuples

Denote $N_g^{\leq,\ell}\binom{\lambda_1,\cdots,\lambda_{s-1}|\ \lambda_s}{\mu_1|\mu_2,\ldots,\mu_n}$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{s-1}$, $1 \leq \ell \leq \lambda_s$ the numbers of monotone transitive tuples $(\sigma_0, \sigma_1, \ldots, \sigma_m)$ such that

- $\sigma_0 = (1\ \cdots\ \lambda_1)(\lambda_1+1\ \cdots\ \lambda_1+\lambda_2)\cdots(d-\lambda_s+1\ \cdots\ d)$;
- $\sigma_m = (r\ d-\lambda_s+\ell)$;
- $\sigma_0\sigma_1\cdots\sigma_m$ is of cyclic type $\boldsymbol{\mu}$; and
- The element d is in the cycle of length μ_1 of the product $\sigma_0\sigma_1\cdots\sigma_m$.

Cut-and-join recursion. Counting tuples.

Cut-and-join for the number of tuples with the fixed σ_0

$$N_g^{\le,\ell}\left(\begin{smallmatrix}\lambda_1,\cdots,\lambda_{s-1}|\ \lambda_s\\ \mu_k|\mu_1,\ldots\widehat{\mu}_k,\ldots,\mu_n\end{smallmatrix}\right) = \Theta(\mu_1-\lambda_s+\ell-1)\sum_{p=1}^{\ell}\Bigg(\sum_{j=2}^{n} N_g^{\le,p}\left(\begin{smallmatrix}\lambda_1,\cdots,\lambda_{s-1}|\ \lambda_s\\ \mu_1+\mu_j|\mu_2,\ldots\widehat{\mu}_j,\ldots,\mu_n\end{smallmatrix}\right)$$

$$+\sum_{\alpha+\beta=\mu_1}\beta N_{g-1}^{\le,p}\left(\begin{smallmatrix}\lambda_1,\cdots,\lambda_{s-1}|\ \lambda_s\\ \alpha|\beta,\mu_2,\ldots,\mu_n\end{smallmatrix}\right)+\sum_{\substack{\alpha+\beta=\mu_1\\ g_1+g_2=g\\ K\sqcup L=\{1,\ldots,s-1\}\\ I\sqcup J=\{2,\ldots,n\}}}\beta N_{g_1}^{\le,p}\left(\begin{smallmatrix}\boldsymbol{\lambda}_K|\lambda_s\\ \alpha|\boldsymbol{\mu}_I\end{smallmatrix}\right)N_{g_2}^{\le}\left(\begin{smallmatrix}\boldsymbol{\lambda}_L\\ \beta,\boldsymbol{\mu}_J\end{smallmatrix}\right)\Bigg)$$

where Θ stands for the Heaviside step function.

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Initialization

$$N_0^{\leq,\ell}\left(\begin{smallmatrix}|\lambda\\ \mu|\end{smallmatrix}\right)=\delta_{\ell,1}\delta_{\lambda,\mu}$$

Free energies

Monotone Hurwitz numbers via the number of tuples

$$H^{\le}_{g,n}\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}} = Z_{\boldsymbol{\lambda}} N^{\le}_{g}\binom{\boldsymbol{\lambda}}{\boldsymbol{\mu}}$$

where $Z_{\boldsymbol{\lambda}}$ is the number of element of $S_{|\boldsymbol{\lambda}|}$ of cyclic type $\boldsymbol{\lambda}$.

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Free energies

$$F^{\leq,(1)}_{g,n} = \frac{1}{n!} \sum_{\substack{\lambda_1 \leq \ldots \leq \lambda_s \\ \mu_1,\cdots,\mu_n}} H^{\leq}_{g,n}\left(\begin{smallmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{smallmatrix}\right) w_{\boldsymbol{\lambda}} p_{\boldsymbol{\mu}}$$

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Kazarian-Zograf operator

$$\delta = \sum_{i=1}^{\infty} i x^{i-1} \frac{\partial}{\partial p_i}$$

$$U^{\le,(1)}_{g,n} = \delta F^{\le,(1)}_{g,n}$$

System of master equations. (0,1)-part.

Suppose that no cycle of $\boldsymbol{\lambda}$ has length more then a. Then the cut-and-join equations for $(g,n)=(0,1)$ are equivalent to the following system of equations on the free energies.

$$xU_{0,1}^{\leq,(\ell)} - xU_{0,1}^{\leq,(\ell+1)} = x^2 U_{0,1}^{\leq,(1)} U_{0,1}^{\leq,(k)} + w_k x^k \quad \text{for } \ell \in \{1,\dots,a\},$$

where the functions $U_{0,1}^{(\ell)}$ with $\ell > 1$ are constructed from the numbers $N_g^{\leq,\ell}\left(\begin{smallmatrix}\lambda_1,\cdots,\lambda_{s-1} | & \lambda_s \\ \mu_1 | \mu_2,\dots,\mu_n & \end{smallmatrix}\right)$.

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Proposition. Suppose that $w_i = 0$ for $i > a$. The substitution $xU_{0,1}^{(1)} = \sum_{k=1}^{a} w_k z^k$, $x = z(1 - \sum_{k=1}^{a} w_k z^k)$ allows to solve the system.

Use Lagrange inversion theorem to recover the numbers.

System of master equations in general case

For $(g,n) \neq (0,1)$ the cut-and-join recursion is equivalent to the following system of master equations:

$$xU^{\leq,(\ell)}_{g,n} - xU^{\leq,(\ell+1)}_{g,n} = x^2 \delta U^{\leq,(\ell)}_{g-1,n+1} + x^2 \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+1}} U^{\leq,(1)}_{g_1,n_1} U^{\leq,(\ell)}_{g_2,n_2}$$

$$+ \delta_y^{-1} d_y \left(\frac{xy}{x-y} U^{\leq,(\ell)}_{g,n-1}(x) - \frac{x^\ell y^{2-\ell}}{x-y} U^{\leq,(\ell)}_{g,n-1}(y) \right).$$

Examples of computation

For $z = z(x), s = s(y)$

$$U_{0,2}^{(1)} = \delta_y^{-1} d_y \left(\frac{1}{x'(z)} \frac{1}{z-s} - \frac{1}{x-y} \right);$$

$$U_{1,1}^{(1)} = \sum_{\ell=1}^{a} \sum_{i=0}^{\ell} \sum_{j=1}^{a} \frac{z^i w_{i+j-1} z^{j-2}}{x'(z)^4} \left(z^2 \frac{x'''(z)}{6} - z^2 \frac{(x''(z))^2}{4x'(z)} + jz \frac{x''(z)}{2} - j(j-1) \frac{x'(z)}{2} \right).$$

Conjecture

For any $(g,n) \neq (0,1),(0,2)$ the correlation functions $\delta_{x_1} \ldots \delta_{x_{n-1}} U_{g,n}^{(1)}$ belong to the space $\bigotimes_{i=1}^{n} O(z_i)$, where $O(z_i)$ is the space of meromorphic functions in the variable z_i having poles in the roots of $x'(z_i)$ only.