Random tensors, a “functional integral” point of view

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But what measure should one use over the random discretizations?

Matrix and tensor models

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Field theories with no kinetic term (probability measures) invariant under conjugation by the unitary group $\mathcal{U}(N)$

- "path integrals" for matrix (tensor) "fields"
- Feynman graphs dual to discretized D-dimensional spaces
- the weight of a discretization is fixed by the Feynman rules: canonical measures over random discretizations.

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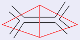
- bonus: the perturbative expansion can be reorganized in powers of $1/N$.
- the perturbative series diverges, but the series at fixed order in $1/N$ converges.

From Matrix to Tensor Models

Invariant action for a matrix M_{ab}

ribbon graphs $\leftrightarrow$ surfaces



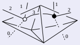
$g(\mathcal{G}) \geq 0$ genus

$1/N$ expansion $A(\mathcal{G}) = N^{2-2g(\mathcal{G})}$

leading order: $g(\mathcal{G}) = 0$, spheres.

Invariant action for a complex, generic tensor $T_{a^1 \dots a^D}$

colored graphs $\leftrightarrow$ D dimensional spaces



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Introduction

Tensor Models

The quartic model

The continuum limit is a phase transition

Tensor invariants as Edge Colored Graphs

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Building blocks: tensors with no symmetry transforming as

$$T'_{b^1\dots b^D} = \sum U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1\dots a^D} \ , \quad \bar{T}'_{p^1\dots p^D} = \sum \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \bar{T}_{q^1\dots q^D}$$

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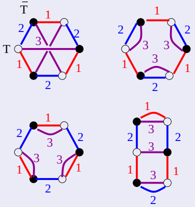
Invariants: colored graphs

$$\mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) = \sum \prod_{v} T_{a^1_v\dots a^D_v} \prod_{\bar{v}} \bar{T}_{q^1_{\bar{v}}\dots q^D_{\bar{v}}} \prod_{c=1}^{D} \prod_{e^c=(w,\bar{w})} \delta_{a^c_w q^c_{\bar{w}}}$$



- White (black) vertices for T ($\bar{T}$).
- Edges for $\delta_{a^c q^c}$ colored by c, the position of the index.

Invariant Actions for Tensor Models

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The most general single trace tensor model

$$S(T,\bar{T}) = \frac{1}{\lambda} \sum T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \prod_{c=1}^{D} \delta_{a^c q^c} + \sum_{\mathcal{B}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(\bar{T},T)$$

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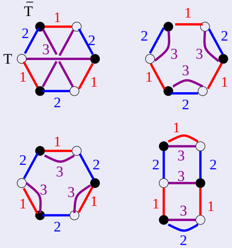

$$\int_{\bar{T},T} e^{-N^{D-1}\left(\frac{1}{\lambda}\sum T_{a^1\dots a^D}\,\bar{T}_{q^1\dots q^D}\prod_{c=1}^{D}\delta_{a^c q^c}\right)}\;\mathrm{Tr}_{\mathcal{B}_1}(\bar{T},T)\mathrm{Tr}_{\mathcal{B}_2}(\bar{T},T)\dots$$

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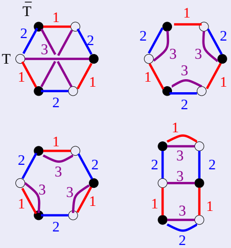

$$\int_{\bar T, T} \qquad e^{-N^{D-1}\left(\frac{1}{\lambda}\sum T_{a^1\dots a^D} \bar T_{q^1 \dots q^D} \prod_{c=1}^{D} \delta_{a^c q^c}\right)}$$

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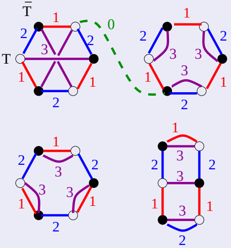

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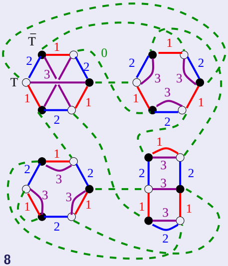

Graphs $\mathcal{G}$ with $D+1$ colors.

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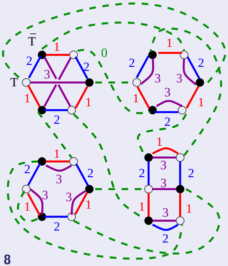

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Represent triangulated D dimensional spaces.

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- $\Big\langle \mathrm{Tr}_{\mathcal{B}}\Big\rangle$: $\mathcal{B}$ to vacuum amplitude
- $\Big\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\Big\rangle_c = \Big\langle \mathrm{Tr}_{\mathcal{B}_1}\mathrm{Tr}_{\mathcal{B}_2}\Big\rangle - \Big\langle \mathrm{Tr}_{\mathcal{B}_1}\Big\rangle\Big\langle \mathrm{Tr}_{\mathcal{B}_2}\Big\rangle$: transition amplitude between the boundary states $\mathcal{B}_1$ and $\mathcal{B}_2$

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Set $t_{\mathcal{B}} \sim (-1) N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})}$ and expand in Feynman graphs $\ln Z_{\mathrm{TM}}(\lambda; N) = \sum_G A^G(\lambda; N)$:

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$$\sum_{\text{topologies}} \int [\mathcal{D}g]\; e^{-\frac{1}{16\pi G}\int d^Dx\sqrt{g}(2\Lambda - R)} \to \sum_{\substack{\text{Triangulations}\\ \text{edge length } a}} e^{-S^{\text{discr.}}_{\text{EH}}(G,\Lambda;a)} = \frac{1}{N^D}\ln Z_{\rm TM}(\lambda;N) ,$$

$$\frac{G}{a^{D-2}} \equiv \tilde G = c_1 \frac{1}{\ln N} , \qquad \Lambda a^2 \equiv \tilde\Lambda = c_2\, \tilde G\, \ln\left(\frac{1}{\lambda}\right) + c_3 , \qquad c_1, c_2, c_3 > 0 \; (\sim O(1)) .$$

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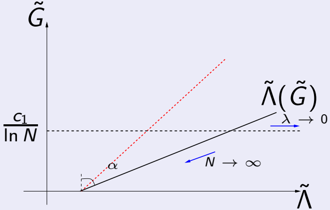

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The simplest quartic invariants correspond to "melonic" graphs with four vertices $\mathcal{B}^{(4),c}$

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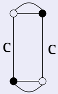

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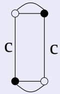

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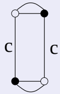

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The simplest observable:

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3) non perturbative: $K_2 = \frac{(1+4Dg)^{\frac{1}{2}} - 1}{2Dg} + \dots + \mathcal{R}_N^{(p)}(g)$, $|\mathcal{R}_N^{(p)}(g)| \leq \frac{1}{N^{p(D-2)}} \frac{|g|^p}{\left(\cos \frac{\arg g}{2}\right)^{2p+2}} \, p! \; A \; B^p$ analytic in $|g| \leq \frac{1}{4D} \left(\cos \frac{\arg g}{2}\right)^2$.

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5) double scaling $(D = 3, 4, 5)$: $g = -\frac{1}{4D} + \frac{x}{N^{D-2}}$, $K_2 = N^{1-\frac{D}{2}} \sum_{p\ge 0} \frac{c_p}{x^{p-\frac{1}{2}}} + \mathit{Rest}$, $\mathit{Rest} < N^{1-D/2}$.
Send $N \to \infty$, $g \to -\frac{1}{4D}$ while keeping x fixed: explore terms subleading in $1/N$.

Continuum limit

Continuum limit

critical point $\Leftrightarrow$ continuum limit

Continuum limit

critical point ⇔ continuum limit

$\lim_{N\to\infty} K_2$ is the sum of an infinite family of triangulations of the sphere ("melons"):

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This continuum limit is a phase transition associated to a symmetry breaking in the tensor model

Phase transitions in field theory

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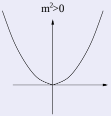

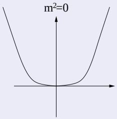

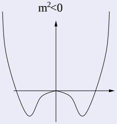

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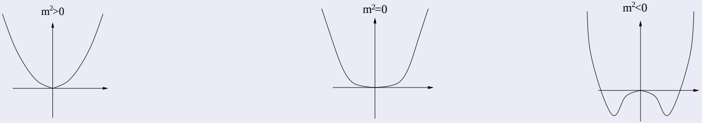

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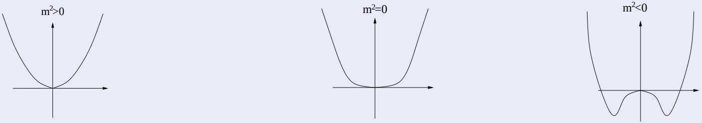

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$$S_{broken} \sim \left(1 + \frac{\rho}{v}\right)^2 \partial\theta\partial\theta + \partial\rho\partial\rho + 2|m^2|\rho^2 + 2|m^2|\rho^3 + \frac{g}{2}\rho^4$$

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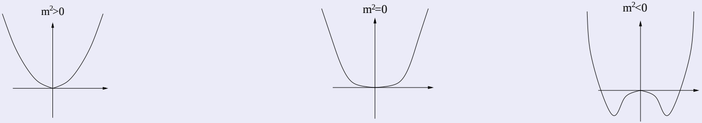

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Phase transition: zero eigenvalue of the "mass matrix"

$$\frac{\delta^2 S^{\text{notkinetic}}}{\delta\bar{\phi}\delta\phi}\Big|_{\bar{\phi}=\phi=0} = m^2 = 0$$

The intermediate field representation

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A Hubbard Stratanovich transformation leads to a coupled multi-matrix model

$$Z(g) = \int \left(\prod_c [dH^c] \right) \; e^{-\frac{1}{2}\sum_c N^{D-1} \mathrm{Tr}_c[H^c H^c] - \mathrm{Tr}_{\mathcal{D}} \left\{ \ln \left[\mathbf{1}^{\otimes \mathcal{D}} - \imath \sqrt{g} \sum_c H^c \otimes \mathbf{1}^{\otimes (\mathcal{D} \setminus c)} \right] \right\}} \; ,$$

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Mass matrix for the fluctuation: $N^{D-1}(1-A^2)\left(\delta^{cc'}\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{DN}\delta_{\alpha\beta}\delta_{\gamma\delta}\right) + N^{D-1}(1-DA^2)\left(\frac{1}{DN}\delta_{\alpha\beta}\delta_{\gamma\delta}\right)$

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At criticality $g = (-4D)^{-1}$, $A^2 = \frac{1}{D}$ the effective mass matrix develops a flat direction $\Rightarrow$ Symmetry breaking!

Conclusion

Tensor models generalize matrix models in higher dimensions and generate random D-dimensional spaces.

Like matrix models, tensor models:

- admit a $1/N$ expansion
- triangulations (graphs) with spherical topology dominate in the large N limit
- exhibit a critical behavior and a continuum limit
- admit a double scaling limit

Unlike in matrix models, in tensor models:

- the number of invariants at fixed order in T and $\bar{T}$ is very large
- the double scaling limit is summable for $D = 3, 4, 5$
- the continuum limit is a phase transition associated to a symmetry breaking