Determinantal formula for the general isomonodromic tau-function

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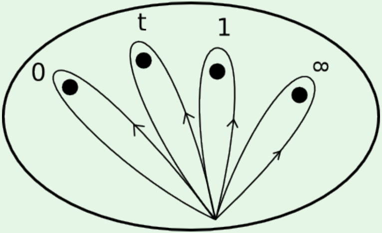
Isomonodromic deformations

Fuchsian system

$$\frac{d}{dz}\Phi(z) = \sum_{\nu=1}^{n} \frac{A_\nu}{z - z_\nu}\Phi(z) = A(z)\Phi(z) \qquad \sum_{\nu} A_\nu = 0$$

Monodromies



$$\gamma_\nu : \Phi(z) \mapsto \Phi(z)M_\nu$$

$$\Phi(z) = (1 + O(z - z_\nu))\,(z - z_\nu)^{A_\nu} C_\nu$$

$$M_\nu = C_\nu^{-1} e^{2\pi i A_\nu} C_\nu \sim e^{2\pi i A_\nu}$$

Isomonodromic deformations

Gauge transformation

$$\Phi(z) \mapsto \left(1 + \epsilon \frac{A_\nu}{z - z_\nu}\right) \Phi(z)$$

$$A(z) \mapsto A(z) + \epsilon \frac{A_\nu}{(z - z_\nu)^2} - \epsilon \left[\frac{A_\nu}{z - z_\nu}, A(z)\right]$$

$$z_\nu \mapsto z_\nu + \epsilon, \quad A_{\mu \neq \nu} \mapsto A_\mu + \epsilon \frac{[A_\nu, A_\mu]}{z_\nu - z_\mu}, \quad A_\nu \mapsto A_\nu - \epsilon \sum_{\mu \neq \nu} \frac{[A_\nu, A_\mu]}{z_\nu - z_\mu}$$

Schlesinger system

$$\frac{\partial A_\mu}{\partial z_\nu} = \frac{[A_\mu, A_\nu]}{z_\mu - z_\nu}, \qquad \frac{\partial A_\nu}{\partial z_\nu} = - \sum_{\mu \neq \nu} \frac{[A_\mu, A_\nu]}{z_\mu - z_\nu}$$

τ-function

$$\frac{\partial}{\partial z_i} \log \tau(\{z_\nu\}) = \frac{1}{2} \operatorname{res}_{z_j} \operatorname{tr} A(z)^2$$

Painlevé equtions

$A(z)$ — 2×2 matrix, 4 regular singular points

$$A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$\sigma(t) = t(t-1)\frac{d}{dt}\log\tau = (t-1)\mathrm{tr}\,A_0A_t + t\,\mathrm{tr}\,A_tA_1$$

Painlevé VI

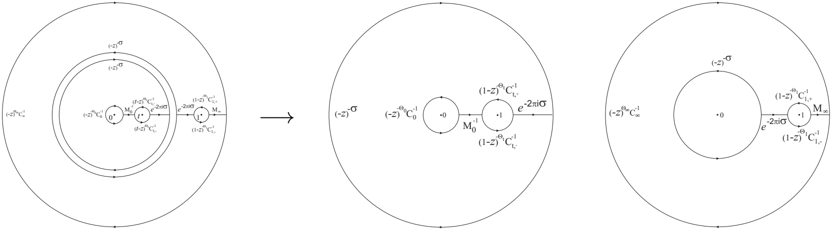
$$\left(t(t-1)\sigma''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

Solutions are proved to be highly transcendental, though the general solution is given explicitly in terms of series over two Young diagrams (O. Gamayn, N. Iorgov, O. Lisovyy [1207.0787]).

$$\tau(t) = t^{\#}\sum_n t^{(\sigma+n)^2}\sum_{Y1,Y2} t^{|Y_1|+|Y_2|}Z_n(Y_1,Y_2,\sigma,\{\theta_i\})$$

Fredholm determinant solution (for 4 points)

Decomposition of the 4-point problem into two 3-point problems



Explicit form of the matrix kernel

$$\tau_{\mathrm{JMU}}(t) = t^{\#} \det(1 - U), \qquad U = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}\left(H^2(S^1) \otimes \mathbb{C}^N\right),$$

where $\mathsf{a} : H^2(S^1)_+ \otimes \mathbb{C}^N \to H^2(S^1)_- \otimes \mathbb{C}^N$, $\quad \mathsf{d} : H^2(S^1)_- \otimes \mathbb{C}^N \to H^2(S^1)_+ \otimes \mathbb{C}^N$

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}(z, z') \, g(z') \, dz', \qquad \mathsf{a}(z, z') = \frac{\Phi^{[R]}(z) \, \Phi^{[R]}(z')^{-1} - 1}{z - z'},$$

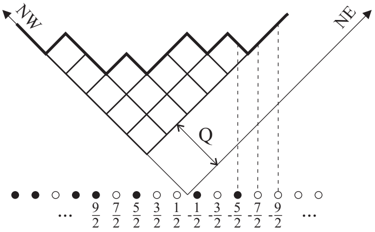
$$(\mathsf{d}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}(z, z') \, g(z') \, dz', \qquad \mathsf{d}(z, z') = \frac{1 - \Phi^{[L]}(z) \, \Phi^{[L]}(z')^{-1}}{z - z'}.$$

Combinatorial expansion in the Fourier basis

$$\tau = \det(1-U) = \sum_{k=0}^{\infty} \sum_{\substack{\mathcal{I}\in 2^{\mathbb{Z}} \\ |\mathcal{I}|=k}} (-1)^k \det U_{\mathcal{I}}^{\mathcal{I}} = \sum_{k=0}^{\infty} \sum_{|I|=|J|=k} (-1)^{\#} \det \mathsf{a}_J^I \cdot \det \mathsf{d}_I^J$$

$$\mathcal{I} = I \sqcup J,\; I = \bigsqcup_{\alpha=1}^{N} I_\alpha,\; J = \bigsqcup_{\alpha=1}^{N} J_\alpha$$

Each pair (I_α, J_α) is encoded by *Maya diagram*, or *charged Young diagram*



$$I = \{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, \tfrac{13}{2}\}, \quad J = \{-\tfrac{1}{2}, -\tfrac{5}{2}\}$$

Therefore

$$\tau = \sum_{k=0}^{\infty} \sum_{|I|=|J|=k} \det \mathsf{a}_J^I \cdot \det \mathsf{d}_I^J = \sum_{\substack{Q_1,\ldots Q_N \\ \sum_\alpha Q_\alpha = 0}} \sum_{Y_1,\ldots Y_N} t^{\frac{1}{2}(\vec{Q}+\vec{\sigma})^2} t^{\sum_\alpha |Y_\alpha|} Z^{\emptyset,0}_{\vec{Y},\vec{Q}}\left(\mathcal{T}^{[L]}\right) Z^{\vec{Y},\vec{Q}}_{\emptyset,0}\left(\mathcal{T}^{[R]}\right)$$

Nekrasov functions (proved rigorously for $N = 2$, generalization is straightforward)

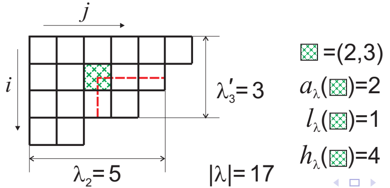
For the special spectral type of A_1: $A_1 \sim \mathrm{diag}(a, 0, \dots, 0)$ solutions Φ's are given by ${}_NF_{N-1}$ and Z can be computed:

$$Z_{\vec{Y},\vec{Q}}^{\vec{Y}',\vec{Q}'}(\mathcal{T}) = \det_{IJ} a_j^i = \frac{\prod_{\alpha,\beta}^N C\left(\sigma'_\alpha - \sigma_\beta \middle| Q'_\alpha, Q_\beta\right)}{\prod_{\alpha<\beta}^N C(\sigma'_\alpha - \sigma'_\beta|Q'_\alpha, Q'_\beta)\, C(\sigma_\alpha - \sigma_\beta|Q_\alpha, Q_\beta)} \frac{e^{i\delta\vec{\eta}'\cdot\vec{Q}' + i\delta\vec{\eta}\cdot\vec{Q}}}{\prod_\alpha^N |Z_{\text{bif}}(0|Y_\alpha, Y_\alpha)|^{\frac{1}{2}} |Z_{\text{bif}}(0|Y'_\alpha, Y'_\alpha)|^{\frac{1}{2}}} \times$$

$$\times \frac{\prod_{\alpha,\beta}^N Z_{\text{bif}}\left(\sigma'_\alpha + Q'_\alpha - \sigma_\beta - Q_\beta \middle| Y'_\alpha, Y_\beta\right)}{\prod_{\alpha<\beta}^N Z_{\text{bif}}(\sigma'_\alpha + Q'_\alpha - \sigma'_\beta - Q'_\beta|Y'_\alpha, Y'_\beta)\, Z_{\text{bif}}\left(\sigma_\alpha + Q_\alpha - \sigma_\beta - Q_\beta \middle| Y_\alpha, Y_\beta\right)}$$

$$C(\nu|Q', Q) \equiv C(\nu|Q' - Q) = \frac{G(1 + \nu + Q' - Q)}{G(1+\nu)\,\Gamma(1+\nu)^{Q'-Q}}$$

$$Z_{\text{bif}}(\nu|Y', Y) := \prod_{\square\in Y'} (\nu + 1 + a_{Y'}(\square) + l_Y(\square)) \prod_{\square\in Y} (\nu - 1 - a_Y(\square) - l_{Y'}(\square))$$



Technical details

Computation of the Fourier components of the kernels

$$a(z,z') = \frac{\Phi(z)\Phi(z')^{-1} - 1}{z - z'} = \sum_{pq\alpha\beta} a^{p,\alpha}_{-q,\beta} z^{-\frac{1}{2}+p+\sigma_\alpha} z'^{-\frac{1}{2}+q-\sigma_\beta}$$

$$\left(z\frac{d}{dz} + z'\frac{d}{dz'} + 1 \right) a(z,z') = -\frac{\Phi(z)}{z-1} A_1 \frac{\Phi(z')^{-1}}{z'-1} = \sum_{k=1}^{\mathrm{rk} A_1} f^{(k)}(z) g^{(k)}(z')$$

$$\left(z\frac{d}{dz} + z'\frac{d}{dz'} + 1 \right) a(z,z') = \sum_{pq\alpha\beta} (p - q + \sigma_\alpha + \sigma_\beta) a^{p,\alpha}_{-q,\beta} z^{-\frac{1}{2}+p+\sigma_\alpha} z'^{-\frac{1}{2}+q-\sigma_\beta}$$

Cauchy matrix (for $\mathrm{rk} A_1 = 1$)

$$a^{p,\alpha}_{-q,\beta} = \frac{f_{\alpha,p} g_{\beta,q}}{p + q - \sigma_\alpha - \sigma_\beta}$$

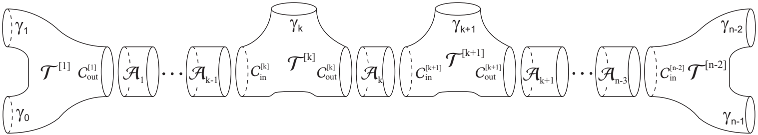
$$\det \frac{1}{x_i - y_j} = \frac{\prod_{i<j} (x_i - x_j)(y_j - y_i)}{\prod_{ij} (x_i - y_j)}$$

Nekrasov functions – result of the cancellation of numerator and denominator.

Remarks

- Obtained formula says that $\tau(t) = \langle \mathcal{O}_0(0)\mathcal{O}_t(t)\mathcal{O}_1(1)\mathcal{O}_\infty(\infty)\rangle$ — correlator in CFT with W_N symmetry.
- Everything is done for arbitrary number of points with the help of appropriate pants decomposition.
- Isomonodromic tau-function exists always, whereas the general W_N conformal block does not. So our formula allows to give an alternative definition for conformal block.

Namely, $\boxed{Z_{\vec{Y},\vec{Q}}^{\vec{Y}',\vec{Q}'}(\mathcal{T})}$ in the case when $\mathcal{T}$ does not correspond to hypergeometric case should be identified with matrix element of the general vertex operator.



Thank you for your attention!