$2d$ quantum gravity with massive matter

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Motivations

$2d$ quantum gravity is useful for:

- toy model for $4d$ quantum gravity
- spontaneous dimensional reduction [1605.05694, Carlip]
- (non-)critical string theories

Real-world requires massive matter

$\rightarrow$ go beyond Liouville action coupled to conformal matter

Functional integration over geometries

- Conformal gauge

$$g = \mathrm{e}^{2\phi} g_0$$

ϕ Liouville mode, g_0 (fixed) background metric

- Partition function

$$Z[\phi] = \mathrm{e}^{-S_{\text{grav}}[g_0,\phi]}\, Z_m[g_0], \quad S_{\text{grav}} = -\ln \frac{Z_m[\mathrm{e}^{2\phi} g_0]}{Z_m[g_0]}$$

$Z_m[g]$ matter partition function

- Typically [1112.1352, Ferrari-Klevtsov-Zelditch]

$$S_{\text{grav}} = S_\mu + \frac{c}{6} S_L + \beta^2 S_M + \cdots$$

S_μ cosmological constant, S_L Liouville action, S_M Mabuchi action

Mabuchi action

- Kähler potential (work at fixed area)

$$\mathrm{e}^{2\phi} = \frac{A}{A_0}\left(1 + \frac{A_0}{2\pi\chi}\,\Delta_0 K\right)$$

$\Delta_0 = g_0^{\mu\nu}\nabla_{0\mu}\nabla_{0\nu}$, χ Euler number

- Mabuchi action (Euclidean) [Mabuchi '86]

$$S_M = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{g_0}\left[-g_0^{\mu\nu}\partial_\mu K\partial_\nu K + \left(\frac{4\pi\chi}{A_0} - R_0\right)K + \frac{4\pi\chi}{A}\,\phi\,\mathrm{e}^{2\phi}\right]$$

- Equation of motion (same as Liouville)

$$R = \frac{4\pi\chi}{A}$$

- Note: ill-defined on the torus/cylinder ($\chi = 0$)

Minisuperspace model

Minisuperspace (background = cylinder)

$$\phi = \phi(t), \qquad K = K(t), \qquad g_0 = \eta$$

Conjectured action (variable area, Lorentzian signature)

$$S_M = -\frac{1}{2} \int \mathrm{d}t \left[\dot{K}^2 - \ddot{K} \ln \left(\frac{\ddot{K}}{4\pi\mu} \right) + \ddot{K} \right], \qquad \mathrm{e}^{2\phi} = \frac{\ddot{K}}{4\pi\mu}$$

μ cosmological constant

Motivations:

- reproduce the features of the full action
- reproduce minisuperspace equation of motion
- 3 different ways to infer this action

Hamiltonian

1. Conjugate momentum to $\dot{K}$

$$P = \frac{\delta S_M}{\delta \ddot{K}} = \frac{1}{2} \ln \left(\frac{\ddot{K}}{4\pi\mu} \right) = \phi$$

2. Canonical transformation

$$P = \phi, \qquad \dot{K} = -\Pi$$

3. Hamiltonian

$$H_M = \frac{\Pi^2}{2} + 2\pi\mu \, \mathrm{e}^{2\phi} = H_L$$

Spectrum

- Canonical quantization [Braaten et al. '84]

$$\hat{H}_M \psi_p = 2p^2\, \psi_p$$

- Wave functions

$$\psi_p(\phi) = \frac{2(\pi\mu)^{-ip}}{\Gamma(-2ip)}\, K_{2ip}(2\sqrt{\pi\mu}\, \mathrm{e}^{\phi}) \sim_{-\infty} \mathrm{e}^{2ip\phi} + R_0(p)\, \mathrm{e}^{-2ip\phi}$$

$p \in \mathbb{R}$: orthonormal set

- 3-point function (limit of DOZZ $b \to 0$)

$$C_0(p_1, p_2, p_3) = \int_{-\infty}^{\infty} \mathrm{d}\phi\, \psi_{p_1}(\phi)\, \mathrm{e}^{-2ip_2\phi} \psi_{p_3}(\phi)$$

$$= (\pi\mu)^{-2\tilde{p}}\, \Gamma(2\tilde{p}) \prod_i \frac{\Gamma((-1)^i 2\tilde{p}_i)}{\Gamma(2p_i)}$$

$$2\tilde{p} = \sum_i p_i, \qquad \tilde{p}_i = \tilde{p} - p_i, \qquad i = 1, 2, 3$$

Conclusion

Main results

- Minisuperspace dynamics of Mabuchi action = Liouville
- Spectrum: $\mathrm{e}^{2ip\phi}$, $p \in \mathbb{R}$

Open problems:

- few physical properties of Mabuchi action known (1-loop string susceptibility)
- Mabuchi should not be a CFT, but zero-mode dynamics is a CFT: how the full dynamics of Mabuchi differs from Liouville?
- find rigorous formulation at variable area and on the torus/cylinder (Kähler formalism not appropriate)
- comparison with matrix models, CDT...