# Non-scalar operators and logarithmic correlation functions for the Potts model in arbitrary dimension

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# Introduction

- **Scale Invariance :** power-law *and* logarithmic correlation
- $c = 0$ CFTs : percolation, disordered systems (IQHE,...)
- Logarithmic minimal models
- Jordan cell in the dilatation operator $L_0$
- Two dimensions collide *J. Cardy 2013*

  $$\langle\phi(0)\phi(r)\rangle = \sum_i \frac{C_i}{r^{2\Delta_i}}, \qquad C_1 \sim -C_2 \to \infty, \quad C_1(\Delta_1 - \Delta_2) \text{ stays finite}$$

  contribution $r^{-2\Delta_1}\log r$
- Insight from discrete symmetries

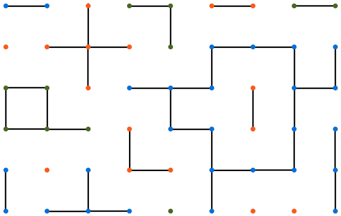
# Potts model

- **$Q$-state Potts model :** discrete $S_Q$ symmetry

$$Z = \sum_{\{\sigma\}} \prod_{(i,j)\in E} e^{K\delta_{\sigma_i,\sigma_j}}, \qquad \sigma = 1, \dots, Q$$

- **Fortuin-Kasteleyn clusters ($Q \in \mathbb{R}$) :**

$$Z = \sum_{A\subseteq E} (e^K - 1)^{|A|} Q^{k(A)}$$



## Operators acting on 1 spin $\sigma$ : *R. Vasseur, J. L. Jacobsen 2014*

- **General form :** $\mathcal{O}(\sigma) = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{\sigma,a}$
- Action of the symmetric group $S_Q$ :

$$(p\mathcal{O})(\sigma) = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{\sigma,p(a)} = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{p^{-1}(\sigma),a}$$

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- **Two irreps :**

$$t(\sigma) = \sum_{a=1}^{Q} \delta_{\sigma,a}, \qquad t_a(\sigma) = \delta_{\sigma,a} - \frac{1}{Q}$$

- **Identity** and **magnetization operator**
- decomposition with Young diagrams $L_Q^{(1)} = [Q] \oplus [Q-1,1]$

# General setup (short version) :

## Operator acting on $N$ spins :

Operators are $Q \times Q \times \ldots = Q^N$ tensors :

$$L_Q^{(N)} = \mathrm{Span}\left\{ \mathcal{O}_{a_1,\ldots,a_N}(\sigma_1,\ldots,\sigma_N) = \prod_{i=1}^{N} \delta_{\sigma_i,a_i} \right\}$$

Action of $p \in S_Q$ : $\mathcal{O}_{\{a_i\}} = \mathcal{O}_{\{p(a_i)\}}$

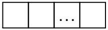
## $S_Q$ symmetry :

- Choose Young tableau $\lambda_Q$ with at least $Q-N$ boxes in the first row
- Compute Young symmetrizer $e_{\lambda_Q}$
- May have to specify $S_N$ representation
- Generate invariant subspace of operators

# Result for $L_Q^{(2)}$, operators acting on $\sigma_1$ and $\sigma_2$ :

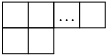
- Subspace with $\sigma_1 = \sigma_2$ isomorphic to $L_Q^{(1)}$
- Symmetric operators under $\sigma_1 \leftrightarrow \sigma_2$

$$t^{[Q]}(\sigma_1, \sigma_2) = \delta_{\sigma_1 \neq \sigma_2}$$



$$t_a^{[Q-1,1],[2]}(\sigma_1, \sigma_2) = \delta_{\sigma_1 \neq \sigma_2} \left( \delta_{\sigma_1, a} + \delta_{\sigma_2, a} - \frac{2}{Q} \right)$$

$$t_{a,b}^{[Q-2,2]}(\sigma_1, \sigma_2) = \delta_{\sigma_1 \neq \sigma_2} \Big( \delta_{\sigma_1, a}\delta_{\sigma_2, b} + \delta_{\sigma_1, b}\delta_{\sigma_2, a} - \frac{1}{Q-2} \left( t_a^{[Q-1,1],[2]} + t_b^{[Q-1,1],[2]} \right) - \frac{2}{Q(Q-1)} t^{[Q]} \Big)$$



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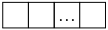

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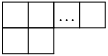
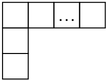


- Anti-symmetric operators under $\sigma_1 \leftrightarrow \sigma_2$

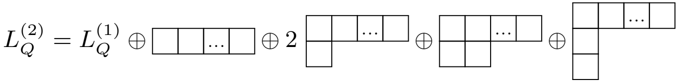
$$t_a^{[Q-1,1],[1,1]}(\sigma_1, \sigma_2) = \delta_{\sigma_1 \neq \sigma_2} \left( \delta_{\sigma_1,a} - \delta_{\sigma_2,a} \right)$$

$$t_{a,b}^{[Q-2,1,1]}(\sigma_1, \sigma_2) = \delta_{\sigma_1 \neq \sigma_2} \Big( \delta_{\sigma_1,a}\delta_{\sigma_2,b} - \delta_{\sigma_1,b}\delta_{\sigma_2,a} - \frac{1}{Q} \left( t_a^{[Q-1,1],[1,1]} - t_b^{[Q-1,1],[1,1]} \right) \Big)$$



# Result for $L_Q^{(2)}$, operators acting on $\sigma_1$ and $\sigma_2$ :

- $L_Q^{(2)}$ **decomposition :**



- Ok with hook formula

$$Q^2 = Q + 1 + 2(Q-1) + \frac{Q(Q-3)}{2} + \frac{(Q-1)(Q-2)}{2}$$

- Subtracted operators correspond to Young diagrams with boxes removed
- **Poles are related to logarithmic features**

## Correlation functions for $N = 1, 2$, some examples

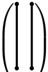
- We can compute correlation functions with FK-clusters

$$\left\langle t_a^{[Q-1,1]} t_b^{[Q-1,1]} \right\rangle = \frac{1}{Q}\left(\delta_{a,b} - \frac{1}{Q}\right) \mathbb{P} \left( \quad \right)$$



- For $N \geq 2$ each operators act on $N$ spins in the same neighbourhood
- Symmetric correlation functions $N = 2$ :

$$\left\langle t_{a,b}^{[Q-2,2]}(0) t_{c,d}^{[Q-2,2]}(r) \right\rangle \propto \left( (2\delta\delta) - \frac{1}{Q-2}(4\delta) + \frac{2}{(Q-2)(Q-1)} \right) \left( \mathbb{P} \left( \quad \right) + \mathbb{P} \left( \quad \right) \right)$$



- Anti-symmetric correlation functions $N = 2$ :

$$\left\langle t_{a,b}^{[Q-2,1,1]}(0) t_{c,d}^{[Q-2,1,1]}(r) \right\rangle \propto \left( (2\delta\delta) - \frac{1}{Q}(4\delta) \right) \left( \mathbb{P} \left( \quad \right) - \mathbb{P} \left( \quad \right) \right)$$

- Correlation functions defined for $Q \in \mathbb{R}$ for finite size

$$\left\langle t_{a,b}^{[Q-2,1,1]}(0) t_{c,d}^{[Q-2,1,1]}(r) \right\rangle \propto \left( (2\delta\delta) - \frac{1}{Q}(4\delta) \right) \frac{\tilde{Z}(Q)}{Z(Q)}$$

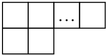
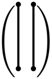
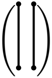
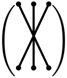
# Scale invariance

- Power-law correlation functions

$$\left\langle t_{a,b}^{[Q-2,1,1]}(0)t_{c,d}^{[Q-2,1,1]}(r)\right\rangle \propto \left((2\delta\delta) - \frac{1}{Q}(4\delta)\right)\frac{1}{r^{2\Delta_{\lambda_2}(Q)}}$$

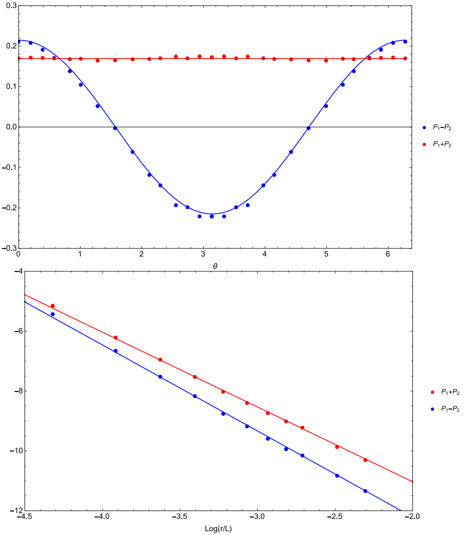
- on the cylinder from Jones-Temperley-Lieb representation theory

$$\Delta = h_{\frac{k}{N},N} + h_{-\frac{k}{N},N}$$

- 4-leg watermelon operator  $\to \mathbb{P}$  $+\ \mathbb{P}$ , $\Delta = 2h_{0,2}$
- Other observables  $\to \mathbb{P}$  $-\ \mathbb{P}$ , $\Delta = h_{-1/2,2} + h_{1/2,2}$
- **Non-scalar**, $s = h_{-1/2,2} - h_{1/2,2} = 1$
- 2d simplified, some configurations are forbidden : $\mathbb{P}$ 

# Numerics

- Transfer matrix methods
- Monte-Carlo simulation



# Logarithmic correlation functions

- How to deal with poles ? **By mixing two operators**
- Percolation : 4-leg watermelon operator diverges at $Q \to 1$

$$t_{a,b}^{[Q-2,2]} = \text{regular part} - \frac{2}{Q(Q-1)}\epsilon, \qquad \epsilon = t^{[Q],[2]} - \left\langle t^{[Q],[2]} \right\rangle$$

$$\phi_{a,b} = t_{a,b}^{[Q-2,2]} + \frac{2}{Q(Q-1)}\epsilon$$

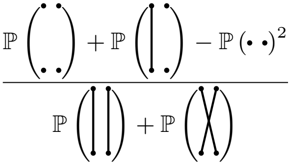
- Correlation functions should be finite $\Delta_\epsilon = \Delta_2$
- Indeed in 2d, $\Delta_\epsilon = \Delta_2 = 5/4$
  Contribution as $Q \to 1$ :

$$\frac{1}{Q-1}\left(\frac{1}{r^{2\Delta_\epsilon(Q)}} - \frac{1}{r^{2\Delta_2(Q)}}\right) \sim \frac{\Delta_\epsilon(Q) - \Delta_2(Q)}{Q-1}\frac{\log r}{r^{2\Delta_2(1)}}$$

- energy and 4-leg watermelon operator mixed in Jordan cell

## Logarithmic correlation functions

- Predict scaling laws on the lattice with universal constant $\delta$



$$\sim \delta r^{-2\Delta_2} \log r$$

- Can be verified with Monte-Carlo
- Whole new set of Jordan cell with non-scalar operators
- Conjecture for the **position of the poles** gives the "full" log structure

# Conclusion

- Classification of operators/LogCFTs in any $d$ with discrete symmetries
- Partial results, other symmetries can be considered
- Jordan-cell of higher rank ?
- Bestiary of interesting geometrical observables