q-deformed Painlevé τ function and q-deformed conformal blocks

M. Bershtein

based on joint papers with A. Shchechkin
arXiv:1608.02566 and 1608.02568

20 October 2016

Painlevé equations

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- Today we concentrate on Painleve III_3

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- Painleve III_3 corresponds to $D_8^{(1)}$ surface and has $D_0^{(1)}$ symmetry.

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  This operator acts $\pi\colon \tau \mapsto \tau_1$, $w \mapsto z/w$.
- Algebraic solution $w = \pm\sqrt{z}$, $\tau(z) = z^{1/16} e^{\pm 4z^{1/2}}$.
  This solution is invariant under Bäcklund transformation π.

Gamayun-Iorgov-Lisovyy formula

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$$z^{1/16}e^{\mp 4\sqrt{z}} = \sum_{n\in\mathbb{Z}} (\mp 1)^n B_n z^{n^2+n/2}\mathcal{F}((1/4+n)^2|z),$$

  where B_n are certain rational numbers. Values $\Delta = (n+1/4)^2$ are conformal dimensions of twist fields [Al. Zamolodchikov 87], [Gavrylenko, Marshakov 15].

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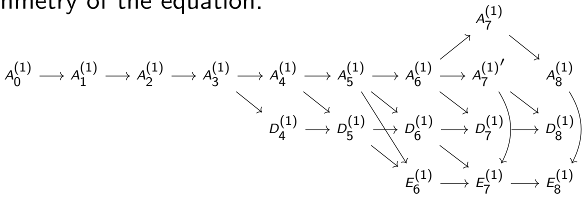
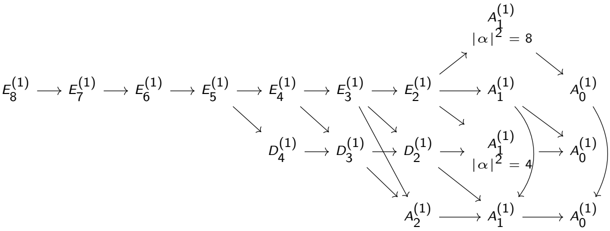


Lattice $R^{\perp}$



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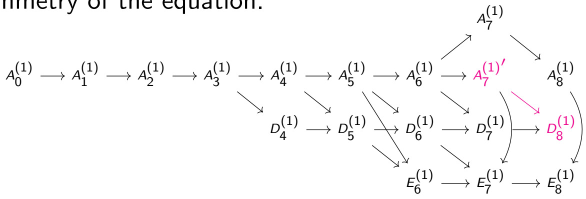

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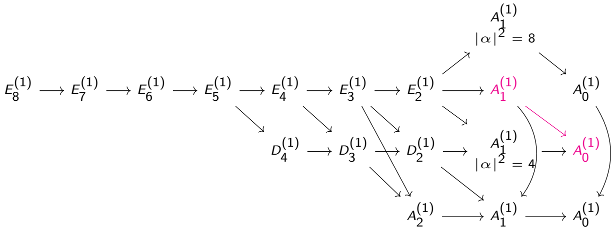

We consider q-difference equation with $R = A_7^{(1)'}$ and $R^{\perp} = A_1^{(1)}$.

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q-difference equation $A_7^{(1)'}/A_1^{(1)}$

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- W is equal to $Dih_4 \ltimes W(A_1^{(1)})$ where Dih_4 is dihedral group of square Group W can be presented by generators s_0, s_1, π_1, π_2 and relations as

  $$s_0^2 = s_1^2 = \pi_1^2 = \pi_2^4 = (\pi_1\pi_2)^2 = 1, \;\; s_1 = \pi_2 s_0 \pi_2^{-1}, \;\; s_0 = \pi_2^2 s_0 \pi_2^{-2} = \pi_1 s_0 \pi_1^{-1},$$

  where s_0, s_1 are Weyl group simple reflections, π_1, π_2 are generators of Dih_4 (symmetry and rotation on 90°).
- The action of W is given in the table

	Z	q	F	G
π_1	Z^{-1}	q^{-1}	$q^{-1}Z^{-1}F$	G^{-1}
π_2	$Z^{-1}q^{-1}$	q	$Z^{-1}G$	F^{-1}
s_1	Z^{-1}	q	$F\frac{(G-1)^2}{(G-Z)^2}$	$Z^{-1}G$
s_0	$Z^{-1}q^{-2}$	q	$q^{-1}Z^{-1}F$	$G\frac{(1-F)^2}{(Zq-F)^2}$

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$$(\overline{Z}, \overline{q}, \overline{F}, \overline{G}) = \left(qZ, q, \frac{(F - qZ)^2}{(F-1)^2 G}, F \right),$$
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- Therefore

$$\overline{G}\underline{G} = \left(\frac{G-Z}{G-1} \right)^2.$$

In the limit $q \to 1$ this equation goes to Painlevé $\mathrm{III}_3(D_8^{(1)})$ equation.

τ functions

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Theorem

Action of generators s_1, π_1, π_2 of group W on $\mathcal{T}_i$, $i = \overline{1,4}$ given below provides a representation of W in the field $\mathbb{C}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, q^{1/4}, Z^{1/4})$.

	Z	q	F	G	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$
π_1	$1/Z$	$1/q$	$F/(qZ)$	$1/G$	$\mathcal{T}_3$	$\mathcal{T}_2$	$\mathcal{T}_1$	$\mathcal{T}_4$
π_2	$1/(qZ)$	q	G/Z	$1/F$	$\mathcal{T}_4$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
s_1	$1/Z$	q	$F\frac{(G-1)^2}{(G-Z)^2}$	G/Z	$\mathcal{T}_1$	$\frac{\mathcal{T}_3^2+Z^{1/2}\mathcal{T}_1^2}{Z^{1/4}\mathcal{T}_4}$	$\mathcal{T}_3$	$\frac{\mathcal{T}_1^2+Z^{1/2}\mathcal{T}_3^2}{Z^{1/4}\mathcal{T}_2}$
s_0	$1/(q^2Z)$	q	$F/(qZ)$	$G\frac{(1-F)^2}{(Zq-F)^2}$	$\frac{\mathcal{T}_4^2+(qZ)^{1/2}\mathcal{T}_2^2}{(qZ)^{1/4}\mathcal{T}_3}$	$\mathcal{T}_2$	$\frac{\mathcal{T}_2^2+(qZ)^{1/2}\mathcal{T}_4^2}{(qZ)^{1/4}\mathcal{T}_1}$	$\mathcal{T}_4$
T	qZ	q	$\frac{(F-qZ)^2}{(-1+F)^2G}$	F	$\mathcal{T}_2$	$\frac{\mathcal{T}_2^2+(qZ)^{1/2}\mathcal{T}_4^2}{(qZ)^{1/4}\mathcal{T}_1}$	$\mathcal{T}_4$	$\frac{\mathcal{T}_4^2+(qZ)^{1/2}\mathcal{T}_2^2}{(qZ)^{1/4}\mathcal{T}_3}$

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π_2	$1/(qZ)$	q	G/Z	$1/F$	$\mathcal{T}_4$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
s_1	$1/Z$	q	$F\frac{(G-1)^2}{(G-Z)^2}$	G/Z	$\mathcal{T}_1$	$\frac{\mathcal{T}_3^2+Z^{1/2}\mathcal{T}_1^2}{Z^{1/4}\mathcal{T}_4}$	$\mathcal{T}_3$	$\frac{\mathcal{T}_1^2+Z^{1/2}\mathcal{T}_3^2}{Z^{1/4}\mathcal{T}_2}$
s_0	$1/(q^2Z)$	q	$F/(qZ)$	$G\frac{(1-F)^2}{(Zq-F)^2}$	$\frac{\mathcal{T}_4^2+(qZ)^{1/2}\mathcal{T}_2^2}{(qZ)^{1/4}\mathcal{T}_3}$	$\mathcal{T}_2$	$\frac{\mathcal{T}_2^2+(qZ)^{1/2}\mathcal{T}_4^2}{(qZ)^{1/4}\mathcal{T}_1}$	$\mathcal{T}_4$
T	qZ	q	$\frac{(F-qZ)^2}{(-1+F)^2G}$	F	$\mathcal{T}_2$	$\frac{\mathcal{T}_2^2+(qZ)^{1/2}\mathcal{T}_4^2}{(qZ)^{1/4}\mathcal{T}_1}$	$\mathcal{T}_4$	$\frac{\mathcal{T}_4^2+(qZ)^{1/2}\mathcal{T}_2^2}{(qZ)^{1/4}\mathcal{T}_3}$

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τ functions

- We follow approach [Tsuda 06] (cf. [Noumi 16])

Theorem

Action of generators s_1, π_1, π_2 of group W on $\mathcal{T}_i$, $i = \overline{1,4}$ given below provides a representation of W in the field $\mathbb{C}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, q^{1/4}, Z^{1/4})$.

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s_1	$1/Z$	q	$F\frac{(G-1)^2}{(G-Z)^2}$	G/Z	$\mathcal{T}_1$	$\frac{\mathcal{T}_3^2+Z^{1/2}\mathcal{T}_1^2}{Z^{1/4}\mathcal{T}_4}$	$\mathcal{T}_3$	$\frac{\mathcal{T}_1^2+Z^{1/2}\mathcal{T}_3^2}{Z^{1/4}\mathcal{T}_2}$
s_0	$1/(q^2Z)$	q	$F/(qZ)$	$G\frac{(1-F)^2}{(Zq-F)^2}$	$\frac{\mathcal{T}_4^2+(qZ)^{1/2}\mathcal{T}_2^2}{(qZ)^{1/4}\mathcal{T}_3}$	$\mathcal{T}_2$	$\frac{\mathcal{T}_2^2+(qZ)^{1/2}\mathcal{T}_4^2}{(qZ)^{1/4}\mathcal{T}_1}$	$\mathcal{T}_4$
T	qZ	q	$\frac{(F-qZ)^2}{(-1+F)^2G}$	F	$\mathcal{T}_2$	$\frac{\mathcal{T}_2^2+(qZ)^{1/2}\mathcal{T}_4^2}{(qZ)^{1/4}\mathcal{T}_1}$	$\mathcal{T}_4$	$\frac{\mathcal{T}_4^2+(qZ)^{1/2}\mathcal{T}_2^2}{(qZ)^{1/4}\mathcal{T}_3}$

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s_1	$1/Z$	q	$F\frac{(G-1)^2}{(G-Z)^2}$	G/Z	$\mathcal{T}_1$	$\frac{\mathcal{T}_3^2+Z^{1/2}\mathcal{T}_1^2}{Z^{1/4}\mathcal{T}_4}$	$\mathcal{T}_3$	$\frac{\mathcal{T}_1^2+Z^{1/2}\mathcal{T}_3^2}{Z^{1/4}\mathcal{T}_2}$
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- Action W can be expressed as composition of mutations in cluster algebra. Therefore it maps $\mathcal{T}_i$ to *Laurent polynomials*.

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$$\boxed{\begin{aligned} Z^{1/4}T(u,s;q|qZ)T(u,s;q|q^{-1}Z) = \mathcal{T}(u,s;q|Z)^2 + \\ Z^{1/2}T(uq,s;q|Z)T(uq^{-1},s;q|Z). \end{aligned}}$$

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Let $|q| \neq 1$ and $u \neq q^n$, $n \in \mathbb{Z}$. Then series (2) converges uniformly and absolutely on every bounded subset of $\mathbb{C}$.

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- $\mathcal{F}(u; q^{-1}, q|Z)/(uq; q, q)_\infty^2$ is topological string partition function for local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry.

Definition

Function $\mathcal{T}(u,s;q|Z)$ given by the formula

$$\mathcal{T}(u,s;q|Z) = R(u;q|Z)\sum_{n\in\mathbb{Z}} s^n C(uq^{2n};q|Z)\frac{\mathcal{F}(uq^{2n};q^{-1},q|Z)}{(uq^{2n+1};q,q)_\infty(u^{-1}q^{-2n-1};q,q)_\infty}$$

is called q-deformed τ function of Painlevé III(D_8) equation if function $C(u;q|Z)$ satisfy equations

$$\begin{aligned} C(uq;q|Z)C(uq^{-1};q|Z)/C(u;q|Z)^2 &= -Z^{1/2} \\ C(uq;q|qZ)C(uq^{-1};q|q^{-1}Z)/C(u;q|Z)^2 &= -uZ^{1/4} \\ C(u;q|qZ)C(u;q|q^{-1}Z)/C(u;q|Z)^2 &= Z^{-1/4} \end{aligned}$$

and function $R(u;q|Z)$ satisfy homogeneous version of these equations (with r.h.s equals 1) and is q^2-periodic in u, i.e. $R(uq^2;q|Z)/R(u;q|Z) = 1$.
If $C(u;q|Z) = C(u^{-1};q|Z)$ and $R(u;q|Z) = R(u^{-1};q|Z)$ then functions $C, R, \mathcal{T}$ are called u-inverse invariant.

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Cf. [Gamayun-Iorgov-Lisovyy 11]

$$\tau(\sigma,s|z)=\sum_{n\in\mathbb{Z}} s^n z^{(\sigma+n)^2}C(\sigma+n)\mathcal{F}((\sigma+n)^2|z)$$

Properties

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- The following examples of $C(u;q|Z)$ satisfy definition

$$C_1(u;q|Z) = \frac{\Gamma((qZ)^{1/4};q^{1/4},q^{1/4})^3}{\big(\Gamma(i(qZu)^{1/4};q^{1/4},q^{1/4})\,\Gamma(i(qZ)^{1/4}u^{-1/4};q^{1/4},q^{1/4})\big)}$$

$$C_c(u;q|Z) = (-1)^{2\left(\frac{\log u}{2\log q}\right)^2}\Gamma(-(qZ)^{1/4};q^{1/4},q^{1/4})\exp\left(\frac{\log^2 u\log Z}{4\log^2 q}\right).$$

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  If the functions C,R are u-inverse invariant then

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Conjecture

Function $\mathcal{T}(u,s;q|Z)$ satisfies bilinear equations

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Theorem

This conjecture is equivalent to

$$\sum_{2n\in\mathbb{Z}} \frac{u^{2n}Z^{2n^2}}{\prod\limits_{\epsilon,\epsilon'=\pm1}(u^{\epsilon}q^{1+2\epsilon' n};q,q)_\infty}\mathcal{F}(uq^{-2n};q^{-1},q|q^{-1}Z)\mathcal{F}(uq^{2n};q^{-1},q|qZ) =$$

$$= (1-Z^{1/2})\sum_{2n\in\mathbb{Z}} \frac{Z^{2n^2}}{\prod\limits_{\epsilon,\epsilon'=\pm1}(u^{\epsilon}q^{1+2\epsilon' n};q,q)_\infty}\mathcal{F}(uq^{-2n};q^{-1},q|Z)\mathcal{F}(uq^{2n};q^{-1},q|Z)$$

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- This conjecture was checked by computer calculation up to Z^3.
- Continuous limit of this relation gives known relation for Virasoro conformal blocks

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Theorem

It follows from Conjecture that for any u-inverse invariant τ *function* $T(u, s; q|Z)$

$$\mathcal{T}(q^{1/2}, \pm 1; q|Z) = \frac{R(q^{1/2}|Z)C(q^{1/2}|Z)}{(q^{3/2}; q, q)_\infty (q^{1/2}; q, q)_\infty} (\mp Z^{1/2} q^{1/2}; q^{1/2}, q^{1/2})_\infty .$$

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- q-difference equation $G(qZ)G(q^{-1}Z) = (G(Z)-Z)^2/(G(Z)-1)^2$ has algebraic solution $G(Z) = \pm\sqrt{Z}$.

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It follows from Conjecture that for any u-inverse invariant τ function $T(u,s;q|Z)$

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  Algebraic solution for Painlevé $D_8^{(1)}$ $\tau(z) \sim z^{1/16}\exp(4\sqrt{z})$,
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  Algebraic solution for q-Painlevé $A_7^{(1')}$ $\mathcal{T}(z) \sim (\mp Z^{1/2}q^{1/2};q^{1/2},q^{1/2})_\infty$,

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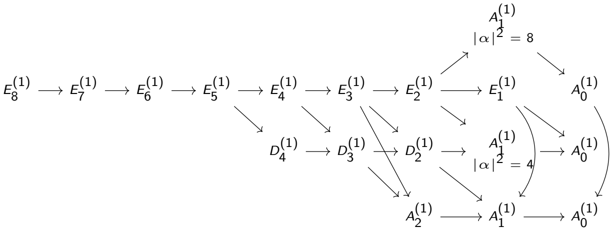

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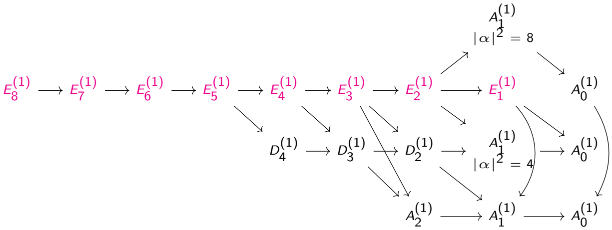

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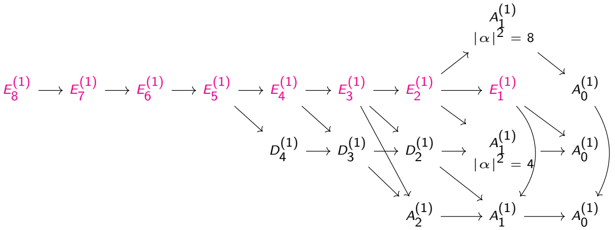

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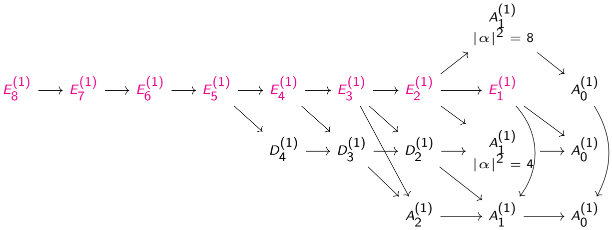

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- for $N \leq 4$ $q \to 1$ limit this relation is known [Gamayun Iorgov Lisovyy 13].
- [Seiberg 96] argued that these gauge theories has E_{N+1} global symmetry. It would be interesting to find a physical interpretation of the affine Weyl groups $E_{N+1}^{(1)}$, the symmetry group of the corresponding discrete Painlevé equations.

Thank you for your attention!

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  $$T\colon \mathcal{T}(u,s;q|Z) \mapsto \mathcal{T}(u,s;q|qZ), \quad \text{and} \quad \pi_2^2\colon \mathcal{T}(u,s;q|Z) \mapsto \mathcal{T}(uq,s;q|Z)$$

  It is natural to ask for the action of whole group W. The remaining transformations are $Z \mapsto Z^{-1}$ and $q \mapsto q^{-1}$. The second transformation is transparent due to $\mathcal{F}(u;q^{-1},q|Z) = \mathcal{F}(u;q,q^{-1}|Z)$.

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- For topological string partition function for local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry there exists fiber-base duality which interchange two factors $\mathbb{P}^1$. In terms of the functions $\mathcal{F}(u;q^{-1},q|Z^{-1})$ this duality has the form

  $$\frac{\mathcal{F}(u;q^{-1},q|Z)}{(uq;q,q)_\infty^2} = \frac{\mathcal{F}(uZ;q^{-1},q|Z^{-1})}{(uZq;q,q)_\infty^2}.$$

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- [Bonelli Grassi Tanzini 16] (using [Grassi Hatsuda Marino 14]) also constructed function, which in the limit $q \to 1$ goes to the $\tau(u,s|z)$. It is interesting to note that they work in different region of q, namely $|q| = 1$ in their paper. There exists a relation between our papers.