Frobenius structures on the deformations of Gepner chiral rings

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Frobenius manifold

Here we review the role of the flat coordinates of the Frobenius manifold for the case of topological Landau-Ginzburg model CFT.

In these models superpotential $W_0[\Phi_1, \ldots, \Phi_n]$ depends on n fundamental chiral fields. These fields generated chiral ring R_0 and we will denote basis in it as Φ_α for $\alpha = 1, \ldots, M$. Here $M = \dim R_0$, first fields with $\alpha = 1, \ldots, n$ will label generators of the ring and $\Phi_1 = 1$ is a unity operator.

The chiral ring R_0 is isomorphic to the ring of the polynomials of x_i

$$R_0 = \mathbb{C}^n[x_1, \ldots, x_n] \Big/ \left\{ \frac{\partial W_0}{\partial x_i} \right\},$$

where $\left\{ \frac{\partial W_0}{\partial x_i} \right\}$ denotes the ideal generated by the partial derivatives of the polynomial $W_0[x_i]$.

Correlators

For computation of the correlation functions of the fields Φ_α and its superpartners $\Phi_\alpha^{(1,1)} = G^-_{-1/2} G^+_{-1/2} \Phi_\alpha$ it is needed and sufficient to know the two-point functions

$$\eta_{\alpha\beta} = \langle \Phi_\alpha \Phi_\beta \rangle,$$

together with the perturbed three-point function

$$C_{\alpha\beta\gamma}(s_1, \ldots, s_M) \stackrel{\text{def}}{=} \langle \Phi_\alpha \Phi_\beta \Phi_\gamma \exp\left(\sum_{\lambda=1}^{M} s_\lambda \int \Phi_\lambda^{(1,1)} \, d^2 z \right) \rangle.$$

It was also shown that $\eta_{\alpha\beta}$ is non-degenerate and s-independent and $C_{\alpha\beta\gamma}(s)$ can be expressed through a prepotential (or free eneregy) $\mathcal{F}$

$$C_{\alpha\beta\gamma} = \frac{\partial^3 \mathcal{F}}{\partial s_\alpha s_\beta s_\gamma}.$$

DVV ring

At last $C^{\gamma}_{\alpha\beta} \stackrel{\text{def}}{=} \eta^{\gamma\delta} C_{\alpha\beta\delta}$ are subject of the equation

$$C^{\rho}_{\alpha\beta} C^{\mu}_{\rho\gamma} = C^{\rho}_{\alpha\gamma} C^{\mu}_{\rho\beta}.$$

At $s_{\alpha} = 0$ this ring coincide with the chiral ring R_0.

These relations together with the evident property $C^{\gamma}_{\alpha\beta} = C^{\gamma}_{\beta\alpha}$ mean that $C^{\gamma}_{\alpha\beta}(s)$ are structure constants for a commutative, associative algebra or a ring R with unity which depends on the parameters $\{s_{\alpha}\}$.

The properties of $\eta_{\alpha\beta}$ and $C^{\gamma}_{\alpha\beta}(s)$ mean indeed that we have the Frobenius manifold structure and s_{μ} are nothing but the flat coordinates on this manifold, i.e. such coordinates in which the Riemann metric $\eta_{\alpha\beta}$ is constant.

DVV ring and the deformed chiral ring

The crucial fact, which makes possible to exactly solve the topological models of such kind, is that the Frobenius manifold, defined by $C_{\alpha\beta\gamma}(s)$ and $\eta_{\alpha\beta}$, coincides with a Frobenis manifold defined by the versal deformations $W(x,t)$ of a superpotential W_0

$$W(x,t) \stackrel{\text{def}}{=} W_0(x) + \sum_{\alpha=1}^{M} t_\alpha e_\alpha(x).$$

Here $\{e_\alpha\}$ is a basis of the ring R_0 (2), $e_1(x) = 1$ is a unity element of R_0. The corresponding ring, defined by $W(x,t)$ as the ring of polynomials of x_i,

$$R_W = \mathbb{C}^n[x_1,\ldots,x_n]\Big/\left\{\frac{\partial W}{\partial x_i}\right\}.$$

The structure constants $\widetilde{C}^{\gamma}_{\alpha\beta}(t)$ of R_W in the basis $\{e_\alpha\}$ are defined by the relations

$$e_\alpha e_\beta = \widetilde{C}^{\gamma}_{\alpha\beta}(t) e_\gamma \mod \{\frac{\partial W}{\partial x_i}\}.$$

Dubrovin axioms

The Riemann metric $g_{\alpha\beta}(t)$ is defined as the Grotendick residue

$$\Omega(x,t) = \lambda(x,t) dx_1 \wedge \cdots \wedge x_n$$

as follows

$$g_{\mu\nu} = \mathrm{Res}_{x=\infty} \frac{e_\mu e_\nu \Omega}{\prod_i \partial W / \partial x_i}.$$

THEOREM, there exist such a differential form $\Omega(x,t)$, that the structure constants $\widetilde{C}^{\gamma}_{\alpha\beta}(t)$ and Riemann metric $g_{\alpha\beta}(t)$ satisfy the Dubrovin Frobenius manifold axioms:

$$\begin{aligned}
\widetilde{C}^{\rho}_{\alpha\beta} \widetilde{C}^{\mu}_{\rho\gamma} &= \widetilde{C}^{\rho}_{\alpha\gamma} \widetilde{C}^{\mu}_{\rho\beta}, \\
R_{\mu\nu\lambda\sigma}[g_{\alpha\beta}] &= 0, \\
\nabla_\sigma \widetilde{C}_{\mu\nu\lambda} &= \nabla_\mu \widetilde{C}_{\sigma\nu\lambda}, \\
\widetilde{C}_{\mu\nu\lambda} &= \widetilde{C}_{\nu\mu\lambda} = \widetilde{C}_{\mu\lambda\nu}.
\end{aligned}$$

The deformation parameters $\{t_\alpha\}$ are coordinates on the Frobenius manifold. The coupling constants s^μ are the flat coordinates functions of the deformation parameters $\{t_\alpha\}$.

The superpotential and basis

We will assume that the superpotential $W_0(x)$ is a quasihomogeneous polynomial associated to an isolated singularity

$$W_0(\Lambda^{\rho_i} x_i) = \Lambda^d W_0(x_i),$$

where integer weights $d = [W_0]$ and $\rho_i = [x_i]$.
In this case we can choose the basis e_α of the ring R_W to be quasihomogeneous . We will denote its weights as $\deg e_\alpha$. The elements of the basis are called relevant, marginal or irrelevant if their weights satisfy correspondingly the relations $\deg e_\alpha < d$, $\deg e_\alpha = d$ or $\deg e_\alpha > d$.

Main Conjecture

It was conjectured that the flat coordinates are given by the following integral expression

$$s_\mu(t) = \sum_{m_\alpha \in \Sigma_\mu} \left(\int_{\gamma_\mu} \exp(W_0(x)) \prod_\alpha e_\alpha^{m_\alpha} \Omega \right) \prod_\alpha \frac{t_\alpha^{m_\alpha}}{m_\alpha!},$$

where Σ_μ is specified by requirement for l.h.s. and r.h.s. of this equation to have the same weights.

The cycles γ_μ form basis for the homology $H_n(\mathbb{C}^n, \mathrm{Re}\, W_0 = -\infty)$ which defined
as $\lim_{L \to +\infty} H_n(\mathbb{C}^n / \{\mathrm{Re}\, W_0 \leq -L\})$ We fix the normalization of

the coordinates by the requirement for the first term of the decomposition to be $s_\mu = t_\mu + \ldots$ and by the requirement that their weights to be equal .

The property of oscialating integrals.

The main point of the computation is the following property of the oscillating integrals

$$\int_\gamma \exp(W_0(x))P_1(x)dx = \int_\gamma \exp(W_0(x))P_2(x)dx,$$

if there exist an $(n-1)$-form U such that

$$P_1(x)dx = P_2(x)dx + D_{W_0}U,$$

where D_{W_0} is Saito differential

$$D_{W_0} = d + dW_0 \wedge .$$

Saito cohomology

The differential D_{W_0} defines the Saito cohomology H^n on the space of n-forms.

The forms $e_\mu dx$ for $\mu = 1, \ldots, M$ can be chosen as a convenient basis in H^n.

Since $e_\mu dx$ form a basis of H^n, any n-form can be decomposed in it. In particular,

$$\prod_\alpha e_\alpha^{k_\alpha} dx = \sum_\mu B_\mu(k) e_\mu dx + D_{W_0} U.$$

From the homogeneity requirements only such elements $e_\mu dx$ of the basis appear in the r.h.s of this equation whose weights are equal to those of the l.h.s or equal module d (resonances).

Their appearence in the oscilating integrals is the reason of arising of the parameters $r_{\mu,\nu}$ in the expressions for s_μ when e_μ and e_ν are in a resonace.

Such parameters arise in result of pairing between the elements $e_\mu dx$ and the cycles γ_μ

$$r_{\mu,\nu} = \int \exp(W_0) e_\nu dx.$$

Resonances and parameters

We choose the homology basis γ_μ to be dual to $e_\mu dx$.
The simplest choice of the dual basis is

$$r_{\mu,\nu} \stackrel{?}{=} \delta_{\mu,\nu}.$$

However, a more general possibility have to be considered. The reason for this is the occurrence of resonances, the cases when the weights satisfy $[s_\mu] - [s_\nu] = 0 \mod d$ and $[s_\mu] \neq [s_\nu]$. We use weaker choice $r_{\mu,\mu} = 1$ for all μ and $r_{\mu,\nu} = 0$ if coordinates s_μ and s_ν are not in resonance. Then we get the expressions for s_μ which depend on extra parameters $r_{\mu,\nu}$.

In the cases of our interest, which is considered below for $\widehat{SU}(3)_3$ there is one resonance $[s_1] - [s_{10}] = 6$.
For the case $\widehat{SU}(3)_4$, , there are two resonances $[s_1] - [s_{14}] = 7$ and $[s_2] - [s_{15}] = 7$. We find that in last case two parameters $r_{1,14}$ and $r_{2,15}$, if they are not assumed to be equal zero, arise in the expressions for the flat coordinates derived from the Conjecture.

The primitive form

From dimensional reasoning the primitive form Ω must be decomposed as

$$\Omega = \sum_{n,l \in \omega} A(n,l) \prod_\alpha e_\alpha^{n_\alpha} t_\alpha^{l_\alpha} dx,$$

where the summation domain ω is defined as

$$\omega : \quad \sum_\alpha (n_\alpha [e_\alpha] + l_\alpha [t_\alpha]) = 0, \quad n_\alpha \geq 0, \quad l_\alpha \geq 0.$$

Using this decomposition one finds

$$s_\mu(t) = \sum_{\substack{m_\alpha \in \Sigma_\mu, \\ n_\alpha, l_\alpha \in \omega}} \left(\int_{\gamma_\mu} \exp(W_0(x)) A(n,l) \prod_\alpha e_\alpha^{m_\alpha + n_\alpha} dx \right) \prod_\alpha \frac{t_\alpha^{m_\alpha + l_\alpha}}{m_\alpha !}.$$

Now, we can give the expression for Σ_μ more explicitly

$$\Sigma_\mu : \quad \sum_\alpha (m_\alpha + l_\alpha)[t_\alpha] = [s_\mu], \quad m_\alpha \geq 0.$$

The explicit expressions for Flat coordinates

Solving the linear equations for the coefficients $B_\mu(k)$ and substituting them into the conjecture we obtain

$$s_\mu(t) = \sum_{\substack{m_\alpha \in \Sigma_\mu, \\ n_\alpha, l_\alpha \in \omega}} A(n,l) B_\mu(m+n) \prod_\alpha \frac{t_\alpha^{m_\alpha + l_\alpha}}{m_\alpha!}.$$

This formula gives expressions for s_μ that depend on the unknown parameters $A(n,l)$ of the primitive form. We find these parameters using the normalisation conditions together with the equation

$$\frac{\partial s_\mu}{\partial t_1} = \delta_{\mu,1}.$$

In such a way we arrive to the explicit expression for flat coordinates. The final answer for the flat coordinates contains no free parameters besides those of $r_{\mu,\nu}$, which correspond to the resonances.

Gepner chiral rings and their deformations

Gepner rings are KS cosets $\hat{U}(N)_k \times \hat{U}(N-1)_1 / \hat{U}(N-1)_{k+1}$ are

$$R_0(N,k) = \mathbb{C}[x_1, ..., x_{N-1}] / \{\partial_1 W_0[x] ... \partial_{N-1} W_0[x]\} \ ,$$

Deformations of these rings possess Frobenius structure because connected with the isolated singularity

$$W_0(x) = \frac{1}{N+k} \sum_{i=1}^{N-1} q_i^{N+k} \ ,$$

where $x_1, ..., x_{N-1}$ are symmetric polynomials of $q_1, ... q_{N-1}$:

$$x_1 = \sum_{i=1}^{N-1} q_i \ , \quad x_2 = \sum_{i<j}^{N-1} q_i q_j \ , \quad ... \ , \quad x_{N-1} = q_1 ... q_{N-1} \ .$$

Schur polynomials are natural homogeneous basis in $R^0(N,k)$.

$$S_\lambda[q_1, ..., q_{N-1}] = \frac{\det q_i^{N+\lambda_j - j}}{\det q_i^{N-j}}$$

Computation of the flat coordinates for $SU(3)_3$ chiral ring

Consider case $k = 3$

$$W_0(x_1, x_2) = \frac{1}{6}(q_1^6 + q_2^6) \ .$$

The basis elements are enumerated by Young diagrams restricted to a rectangle of size $k \times (N-1)$ with $k = 3$, $N = 3$.

$$e_1 \equiv e_{\varnothing} = 1 \ , \quad e_2 \equiv e_{\ydiagram{1}} = q_1 + q_2 \ , \quad e_3 \equiv e_{\ydiagram{1,1}} = q_1 q_2 \ .$$

$$e_4 \equiv e_{\ydiagram{2}} = q_1^2 + q_1 q_2 + q_2^2 \ ,$$

$$e_5 \equiv e_{\ydiagram{2,1}} = q_1 q_2 (q_1 + q_2) \ , \quad e_6 \equiv e_{\ydiagram{3}} = q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3 \ ,$$

$$e_7 \equiv e_{\ydiagram{2,2}} = q_1^2 q_2^2 \ ,$$

$$e_8 \equiv e_{\ydiagram{3,1}} = q_1 q_2 (q_1^2 + q_1 q_2 + q_2^2) \ , \quad e_9 \equiv e_{\ydiagram{3,2}} = q_1^2 q_2^2 (q_1 + q_2) \ ,$$

$$e_{10} \equiv e_{\ydiagram{3,3}} = q_1^3 q_2^3 \ .$$

It is the case when one marginal deformations appears in the ring.

Computation flat coordinates for k=3

Therefore we have to allow the primitive form to be a function of parameter t_{10} whose weght is equal 1. The flat coordinates are given by the following expression

$$s_\mu(t) = \lambda(t_{10}) \sum_{\{m_\lambda\}\in\Sigma_\lambda} \sum_{n=0}^{\infty} C_\mu(m_\lambda|n) \prod_{1\le\lambda<10} \frac{t_\lambda^{m_\lambda}}{m_\lambda!} \frac{t_{10}^n}{n!} ,$$

Coefficients $C_\mu(m_\lambda|n)$ are fixed by the following condition

$$\prod_\lambda e_\lambda^{m_\lambda} e_{10}^n \Omega_0 = \sum_\mu C_\mu(m_\lambda|n) e_\mu \Omega_0 + D_{W_0} U ,$$

$$\Omega_0 = (q_1 - q_2) dq_1 \wedge dq_2 ,$$

and 1-form can be written as

$$U = A(q_1, q_2) dq_1 + B(q_1, q_2) dq_2 ,$$

Since the isolated singularity is homogeneous, the $(n-1)$-form U can be represented as a sum of homogeneous terms $\sum_{l=0}^{L} U_l$ and the equation $C_\mu(\vec{m}|n)$ takes the form

$$dW_0\,U_0 = \prod_\lambda e_\lambda^{m_\lambda}\,dx\;,$$

$$dW_0\,U_l = -dU_{l-1}\;,$$

$$\sum_\mu C_\mu(\vec{m}|n)e_\mu\,dx = -dU_L\;,$$

where $l = 1, ..., L$ and $L = \sum m_\alpha - 1$.

Computation flat coordinates for k=3

More explicitly we find

$$\prod_{\lambda} e_{\lambda}^{m_\lambda} e_{10}^{n}(q_1-q_2) = \sum_{\mu} C_{\mu}(m_\lambda|n) e_{\mu}(q_1-q_2) + \frac{\partial B}{\partial q_1} - \frac{\partial A}{\partial q_2} + A q_2^5 - B q_1^5 \ .$$

We can transform this relation into the set of recurrence relations

$$A = \sum_{s=0}^{L} A_s \ , \quad B = \sum_{s=0}^{L} B_s \ .$$

$$\prod_{\lambda} e_{\lambda}^{m_\lambda}(q_1 - q_2) = A_0 q_2^5 - B_0 q_1^5 \ .$$

$$\frac{\partial A_{s-1}}{\partial q_2} - \frac{\partial B_{s-1}}{\partial q_1} = A_s q_2^5 - B_s q_1^5, \qquad 1 \leq s \leq L \ .$$

$$\sum_{\mu} C_{\mu}(m_\lambda|n) e_{\mu}(q_1, q_2) = \frac{\partial A_L}{\partial q_2} - \frac{\partial B_L}{\partial q_1} \ .$$

These relations allow to reconstruct all the coefficient A_i, B_i and to express $C_\mu(m_\lambda|n)$ in terms A_L, B_L.

Explicit expressions for the flat coordinates for $t_{10} \neq 0$

When $t_{10} \neq 0$ the flat coordinates become to be series of t_{10}. Let us demonstrate the general idea explained in the previous section on the following examples for $k = 3$.
In our first example we compute the contribution to s_1 which contains t_1. It has the following form

$$s_1 = \lambda(t_{10}) f_1(t_{10}) t_1 + \ldots ,$$

where dots stand for possible contributions of other t_ν with $\nu \neq 1$.
Our goal now is to find $f_1(t_{10})$.

$$f_1(t_{10}) = \sum_n C(n) \frac{t_{10}^{2n}}{(2n)!} .$$

As explained above we compute the coefficients $C(n)$ from the equation

Computation C(n)

$$e_1 e_{10}^{2n}\Omega = C(n)\Omega + D_{W_0}U \; .$$

Since $e_1 = 1$, we have to solve

$$e_{10}^{2n}(q_1 - q_2) = C(n)(q_1 - q_2) + \frac{\partial B}{\partial q_1} - \frac{\partial A}{\partial q_2} + Aq_2^5 - Bq_1^5$$

To do this it is convenient derive the following recurrence relation for the coefficients

$$C(n+1) = P(n)C(n) \; .$$

Since $e_{10} = q_1^3 q_2^3$, we get

$$q_1^{6n+6}q_2^{6n+6}(q_1 - q_2) = q_1^{6n}q_2^{6n}(q_1 - q_2) + \dots \; ,$$

which is to be presented in the form

$$q_1^{6n+7}q_2^{6n+6} - q_1^{6n+6}q_2^{6n+7} = Aq_2^5 - Bq_1^5 + \frac{\partial B}{\partial q_1} - \frac{\partial A}{\partial q_2} + \\ + P(n)[q_1^{6n+1}q_2^{6n} - q_1^{6n+6}q_2^{6n+1}] \; .$$

Computation C(n)

The coefficients are easily found $A_0 = -q_1^{6n+6} q_2^{6n+2}$, $B_0 = -q_1^{6n+2} q_2^{6n+6}$. So that

$$\frac{\partial A_0}{\partial q_2} - \frac{\partial B_0}{\partial q_1} = (6n+2)[q_1^{6n+1} q_2^{6n+6} - q_1^{6n+6} q_2^{6n+1}] \,,$$

which in turn is to be equal to

$$(6n+2)[q_1^{6n+1} q_2^{6n+6} - q_1^{6n+6} q_2^{6n+1}] = A_1 q_2^5 - B_1 q_1^5 \,,$$

so that $A_1 = (6n+2) q_1^{6n+1} q_2^{6n+1}$, $B_1 = (6n+2) q_1^{6n+1} q_2^{6n+1}$. The final condition is

$$\frac{\partial A_1}{\partial q_2} - \frac{\partial B_1}{\partial q_1} = (6n+1)(6n+2)[q_1^{6n+1} q_2^{6n} - q_1^{6n} q_2^{6n+1}] = P(n)[q_1^{6n+1} q_2^{6n} - q_1^{6n} q_2^{6n+1}] \,.$$

Hence,

$$P(n) = (6n+1)(6n+2)\ ,$$

and

$$C(n) = \prod_{m=0}^{n-1} (6m+1)(6m+2)\ .$$

We conclude that

$$f_1(t_{10}) = 1 + \sum_{n=1}^{\infty} \prod_{m=0}^{n-1} (6m+1)(6m+2) \frac{t_{10}^{2n}}{(2n)!} = {}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)\ .$$

Taking into account the normalization requirement for s^1 we obtain

$$\lambda(t_{10}) = \frac{1}{f_1(t_{10})} = \frac{1}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)}\ .$$

Computation s^{10}

Our second example demonstrates the computation of the flat coordinate

$$s_{10} = \lambda(t_{10}) t_{10} f_{10}(t_{10}) = \lambda(t_{10}) \sum_{n=1}^{\infty} C(n) \frac{t_{10}^{2n+1}}{(2n+1)!} \ .$$

with $C(0) = 1$. We have the following equations for the coefficients $C(n)$

$$e_{10}^{2n+1} \Omega_0 = C(n) e_{10} \Omega + D_{W_0} U \ ,$$

and

$$C(n+1) = P(n) C(n) \ .$$

Or explicitly,

$$e_{10}^{2n+3}(q_1 - q_2) = P(n) e_{10}^{2n+1}(q_1 - q_2) + A q_2^5 B q_1^5 + \frac{\partial B}{\partial q_1} - \frac{\partial A}{\partial q_2} \ .$$

Computation s_{10}

It follows that

$$U = U_0 + U_1 \ , \quad A = A_0 + A_1 \ , \quad B = B_0 + B_1 \ ,$$

$$\begin{aligned} A_0 &= -q_1^{6n+9} q_2^{6n+4} \ , & B_0 &= -q_1^{6n+4} q_2^{6n+9} \ , \\ A_1 &= (6n+4) q_1^{6n+3} q_2^{6n+5} \ , & B_1 &= (6n+4) q_1^{6n+5} q_2^{6n+3} \ , \end{aligned}$$

and

$$P(n) = (6n+4)(6n+5) \ .$$

We find

$$f_{10}(t_{10}) = 1 + \sum_{n=1}^{\infty} \prod_{m=0}^{n-1} (6m+4)(6m+5) \frac{t_{10}^{2n+1}}{(2n+1)!} \ .$$

we obtain

$$s_{10}(t_{10}) = t_{10} \lambda(t_{10}) \, {}_2F_1 \left(\frac{2}{3}, \frac{5}{6}; \frac{3}{2} \middle| 9t_{10}^2 \right) = t_{10} \frac{{}_2F_1 \left(\frac{2}{3}, \frac{5}{6}; \frac{3}{2} \middle| 9t_{10}^2 \right)}{{}_2F_1 \left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2 \right)} \ .$$

Expressions for other coordinates

$$s_9(t) = t_9 \frac{{}_2F_1\left(\frac{5}{6}, \frac{1}{2}; \frac{1}{2} \middle| 9t_{10}^2\right)}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)},$$

$$s_8(t) = t_8 \frac{{}_2F_1\left(\frac{2}{3}, \frac{1}{2}; \frac{1}{2} \middle| 3t_{10}\right)}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)} + t_9^2 \frac{{}_2F_1\left(\frac{5}{3}, \frac{1}{2}; \frac{1}{2} \middle| 3t_{10}\right)}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)},$$

$$s_7(t) = t_7 \frac{{}_2F_1\left(\frac{2}{3}, \frac{1}{2}; \frac{1}{2} \middle| 9t_{10}^2\right)}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)} + 6t_9^2 t_{10} \frac{{}_2F_1\left(\frac{5}{3}, \frac{1}{2}; \frac{1}{2} \middle| 9t_{10}^2\right)}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)}.$$

These expressions together with formula for Saito primitive form Ω

$$\Omega = \lambda(t_{10})dx = \frac{dx}{{}_2F_1\left(\frac{1}{6}, \frac{1}{3}; \frac{1}{2} \middle| 9t_{10}^2\right)}.$$

give in fact the exact solution for the corresponding TCFT.

Computation of the flat coordinates for $\widehat{SU}(3)_4$ chiral ring

In the topological CFT,which is connected with the deformed chiral ring $\widehat{SU}(3)_4$, the superpotential is

$$W_0(x_1, x_2) = \frac{q_1^7 + q_2^7}{7},$$

where $x_1 = q_1 + q_2$, $x_2 = q_1 q_2$.
We choose the basis of the ring to be Schur polynomials

$$e_1 \equiv e_{\varnothing} = 1\,, \quad e_2 \equiv e_{(1)} = q_1 + q_2\,, \quad e_3 \equiv e_{(1,1)} = q_1 q_2\,.$$

$$e_4 \equiv e_{(2)} = q_1^2 + q_1 q_2 + q_2^2\,, \quad e_5 \equiv e_{(2,1)} = q_1 q_2 (q_1 + q_2)\,,$$

$$e_6 \equiv e_{(3)} = q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3\,, \quad e_7 \equiv e_{(2,2)} = q_1^2 q_2^2\,,$$

$$e_8 \equiv e_{(3,1)} = q_1 q_2 (q_1^2 + q_1 q_2 + q_2^2)\,, \quad e_{10} \equiv e_{(3,2)} = q_1^2 q_2^2 (q_1 + q_2)\,,$$

$$e_9 \equiv e_{(4)} = q_1^4 + q_1^3 q_2 + q_1^2 q_2^2 + q_1 q_2^3 + q_2^4\,,$$

$$e_{11} \equiv e_{(4,1)} = q_1 q_2 (q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3)\,,$$

$$e_{12} \equiv e_{(3,3)} = q_1^3 q_2^3\,, \quad e_{13} \equiv e_{(4,2)} = q_1^2 q_2^2 (q_1^2 + q_1 q_2 + q_2^2)\,,$$

$$e_{14} \equiv e_{(4,3)} = q_1^3 q_2^3 (q_1 + q_2)\,, \quad e_{15} \equiv e_{(4,4)} = q_1^4 q_2^4.$$

Check the Conjecture

To verify our Conjecture we have to compute the flat coordinates by the direct way. We did it perturbatively in overall t up to 4th order in t. The final answers are too lengthy to be presented here. Therefore, we will only outline the main steps of the calculation giving as many details as possible.

The metric on the Frobenius manifold is defined as

$$g_{\mu\nu} = \mathrm{Res}_{x=\infty} \frac{e_\mu e_\nu \Omega}{\prod_i \partial W/\partial x_i}.$$

Instead of computing this residue we will rewrite the metric as

$$g_{\mu\nu} = C^{\lambda}_{\mu\nu}(t) \mathrm{Res}_{x=\infty} \frac{e_\lambda \Omega}{\prod_i \partial W/\partial x_i} = C^{\lambda}_{\mu\nu}(t) h_\lambda(t),$$

where $h_\mu(t)$ are some unknown functions of t.

Solving Dubrovin axioms

These functions can be found from the Frobenius axioms

$$
\begin{aligned}
R_{\mu\nu\lambda\sigma}[g_{\alpha\beta}] &= 0, \\
\nabla_\sigma C_{\mu\nu\lambda} &= \nabla_\mu C_{\sigma\nu\lambda}, \\
C_{\mu\nu\lambda} &= C_{\nu\mu\lambda} = C_{\mu\lambda\nu},
\end{aligned}
$$

where $R_{\mu\nu\lambda\sigma}$ -Riemann curvature, $C_{\mu\nu\lambda}$ is structure constants with index lowered by $g_{\alpha\beta}$.
Finally, one can find flat coordinates from the equation

$$
\frac{\partial^2 s_\mu}{\partial t_\alpha \partial t_\beta} = \Gamma^\gamma_{\alpha\beta} \frac{\partial s_\mu}{\partial t_\gamma}.
$$

Since the metric found from these equations contains one parameter the flat coordinates will also contain a parameter. These results are in perfect agreement up to the fourth order with the conjecture if we impose the constraint $r_{1,14} = r_{2,15}$.

Comparison with M. Saito theorem

Here we want to connect our results with one of M. Saito.
We will relabel weights of basis diagrams as $\sigma_i = \deg e_i / \deg W_0$.
Basis in the ring must be ordered in such a way that $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_M$. It was proved that dimension D of the moduli space of the primitive form (or number of free parameters) is given by the formula

$$D = \#\{(i,j) \,|\, p(i,j) \in \mathbb{Z}_{>0}, i+j < M+1\}$$
$$+\#\left\{(i,j) \,|\, p(i,j) \in \mathbb{Z}_{>0}^{\text{odd}}, i+j = M+1\right\},$$

where $p(i,j) = \sigma_i - \sigma_j$, M is a dimension of the chiral ring, $\mathbb{Z}_{>0}$ are positive integers and $Z_{>0}^{\text{odd}}$ are odd positive integers.

It follows from this Theorem that number modules is one for both cases $\widehat{SU}(3)_k$ with $k=3$ and $k=4$. Hower in our computation $k=4$ we obtain two parameters. If we have to impose on them the constraint $r_{1,14} = r_{2,15}$.

Some results k=4

$$
\begin{aligned}
s_{15} &= t_{15} - (r_{1,14} + r_{2,15})t_{14}t_{15},\\
s_{14} &= t_{14} - r_{1,14}t_{14}^2 - r_{2,15}t_{13}t_{15} - r_{2,15}t_{12}t_{15},\\
s_{13} &= t_{13} + (3 - r_{1,14})t_{13}t_{14} + (3 - r_{2,15})t_{11}t_{15} - r_{2,15}t_{10}t_{15},\\
s_{12} &= t_{12} - r_{1,14}t_{12}t_{14} - r_{2,15}t_{10}t_{15},\\
s_{11} &= t_{11} + t_{13}^2 + 2t_{12}t_{13} + (2 - r_{1,14})t_{11}t_{14} + 2t_{10}t_{14}\\
&\quad - r_{2,15}t_9t_{15} + (2 - r_{2,15})t_8t_{15},\\
s_{10} &= t_{10} + \frac{3t_{13}^2}{2} + 3t_{11}t_{14} - r_{1,14}t_{10}t_{14} + 3t_9t_{15}\\
&\quad - r_{2,15}t_8t_{15} - r_{2,15}t_7t_{15},\\
s_9 &= t_9 + t_{11}t_{13} + t_{10}t_{13} + t_{10}t_{12} - r_{1,14}t_9t_{14} + t_8t_{14} + t_7t_{14}\\
&\quad - r_{2,15}t_6t_{15} + t_5t_{15},\\
s_8 &= t_8 + 2t_{11}t_{13} + 2t_{11}t_{12} + 2t_{10}t_{13} + 2t_9t_{14} + (2 - r_{1,14})t_8t_{14}\\
&\quad + (2 - r_{2,15})t_6t_{15} - r_{2,15}t_5t_{15},\\
s_7 &= t_7 + 3t_{11}t_{13} + 3t_9t_{14} - r_{1,14}t_7t_{14} - r_{2,15}t_5t_{15},
\end{aligned}
$$

Some results k=4

$$\Omega = \Big[1 - r_{1,14}t_{14} + \left(r_{1,14}^2 - 1\right) t_{14}^2 + r_{1,14}r_{2,15}t_{13}t_{15} + \left(r_{1,14}r_{2,15} - 2\right) t_{12}t_{15}\Big] e_1 + \Big[- r_{2,15} + \left(r_{2,15}^2 + r_{1,14}r_{2,15} - 3\right) t_{14}\Big] t_{15}e_2 - 3t_{15}^2 e_3.$$

Conclusion

Thus we have the efficient way to compute the flat coordinates and the primitive form for Frobenius manifolds connected with isolated singularities.

The main natural applications of this result is the computation of the correlators of the models of TCFT and of 2d $\mathcal{W}$ gravity.

Our approach is based on the integral representation for the flat coordinates which is Conjecture in case when besides the relevent deformations there are also the marginal ones. This Conjecture was verified by an independent direct computation.

It would be interesting to generalize this construction on the general $\hat{U}(N)_k \times \hat{U}(N-1)_1/\hat{U}(N-1)_{k+1}$ case and to check whether the Conjecture is correct in cases when Jacobi rings of the isolated singularities include not only relevant and marginal elements but irrelevant ones.

Conclusion

Another possible application is related to the compactifications of the superstring theory on Calabi-Yau manifolds. The flat coordinates of the corresponding Frobenius manifods are related to the periods of holomorphic 3-form evaluated on Calabi-Yau manifolds.

The knowledge of the periods is important because the complete structure of the moduli space of Calabi-Yau manifolds and the corresponding low-energy effective theory that results from $N = 2$ superstring comspactification may be determined in terms of the periods.