4-point correlation numbers in Minimal Gravity and Liouville OPE discrete terms

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Introduction

1. Non-critical String Theory and fluctuating surfaces - A. Polyakov (1981).
2. Minimal Models of Conformal Field Theory - A. Belavin A. Polyakov A. Zamolodchikov (1984)
3. Conformal Bootstrap in Liouville theory - A. Zamolodchikov and Al. Zamolodchikov (1995)
4. 4-point correlation numbers in Minimal Liouville Gravity (MLG) through Higher Equations of Motion (HEM) - A. Belavin and Al. Zamolodchikov (2005)

1. Another approach to 2d gravity (Matrix Models of 2d gravity) - Brezin and Kazakov; Gross and Migdal; Douglas and Shenker; Douglas (1990)
2. Resonance relations in the Lee-Yang series $(2, p)$ MLG - A. Belavin and A. Zamolodchikov (2008)
3. Connection with Frobenius manifolds and fusion rules problem in $(3, p)$ series - A. Belavin, B. Dubrovin and B. Mukhametzhanov (2014)
4. Dual Douglas equation approach for Lee-Yang series - V. Belavin and Yu. Rud (2015)

Results

1. Though these two approaches to 2d quantum gravity are similar, they give somewhat different results, in particular continuous approach doesn't give all 4-point correlation numbers.

2. In the paper we modified and extended the old HEM formula for correlation numbers and checked the result against Douglas equation approaches in the Lee-Yang series of models.

3. We performed various numerical checks which confirm our results.

Minimal Models

Minimal models $\mathcal{M}(p'/p)$ are rational conformal field theories. Physical fields belong to a sum of a finite number of highest weight representations of Virasoro algebra with central charge

$$c_M = 1 - 6(b^{-1} - b)^2,\ b = \sqrt{p'/p}$$

$(p'-1)(p-1)$ primary fields $\Phi_{m,n}(x)$ correspond to highest weight vectors with weigths given by $L_0\Phi_{m,n} = \Delta^M_{m,n}\Phi_{m,n}$.

$$\lambda_{m,n} = \frac{mb + nb^{-1}}{2},$$

$$\Delta^M_{m,n} = \lambda^2_{m,-n} - \lambda^2_{1-1}$$

There is an explicit expression for $\mathbb{C}^i_{jk}$. Physical fields OPE satisfy sl_2-like fusion rules

$$[\Phi_{m_1,n_1}][\Phi_{m_2,n_2}] = \sum_{r=|m_1-m_2|+1,s=|n_1-n_2|+1}^{f(m_1,m_2),f(n_1,n_2)} [\Phi_{r,s}]$$

$f(m,n) = \min(m+n-1,\ 2p' \text{ or } 2p - m - n - 1)$.

One can also consider Generalized Minimal Models with formal fields of arbitrary complex dimension.

As a conformal field theory LFT is an irrational CFT with central charge

$$c_L = 1 + 6Q^2 = 1 + 6(b^{-1} + b)^2$$

and continuous spectrum of primary fields $V_a(x)$ with conformal dimensions

$$\Delta_a^L = a(Q - a).$$

Fields in the spectrum can be identified with $V_{Q/2-iP}$ for real P.

Degenerate fields (i.e. fields whose corresponding Verma module is reducible) are $V_{m,n} = V_{a_{m,n}} \sim V_{Q-a_{m,n}}$ where

$$a_{m,n} = Q/2 - \lambda_{m,n}$$

The basic Liouville operator product expansion:

$$V_{a_1}(x)V_{a_2}(0) = \int' \frac{dP}{4\pi} (x\bar{x})^{\Delta-\Delta_1-\Delta_2} \mathbb{C}_{a_1,a_2}^{Q/2+iP} \left[V_{Q/2+iP}(0)\right] \qquad (1)$$

This OPE is continuous and involves integration over the “momentum” P. The prime on the integral indicates possible discrete terms (the same phenomenon as in the talks by A. Rosso and X. Cao yesterday)

LG is represented as a tensor product of the conformal matter (M), Liouville theory, and the ghost system

$$A_{\mathrm{LG}} = A_{\mathrm{M}} + A_{\mathrm{L}} + A_{\mathrm{G}}$$

with the interaction via the relation for central charge parameters and definition of observables.

$$c_{\mathrm{M}} + c_{\mathrm{L}} + c_{\mathrm{G}} = 0 \tag{2}$$

Central charge balance condition (2) implies the same value of parameter b for MM and LFT.

We are interested in correlators of the composite fields

$$\mathbb{U}_a(x) = \Phi_{a-b}(x) V_a(x), \ \mathbb{W}_a(x) = C(x)\bar{C}(x)\mathbb{U}_a(x), \tag{3}$$

which are one form and scalar field respectively. n-point correlation numbers on the sphere for these observables are

$$I_n(a_1, \ldots, a_n) = \int d^2 z_i \left\langle \prod_{i=4}^{n} \mathbb{U}_{a_i}(z_i) \mathbb{W}_{a_3}(z_3)\, \mathbb{W}_{a_2}(z_2)\, \mathbb{W}_{a_1}(z_1) \right\rangle. \tag{4}$$

HEM formula

For four points if one of the matter fields is degenerate, HEM allows to reduce the moduli integral in general expression (4) to boundary integrals:

$$I_4(a_1 = a_{m,-n}, a_2, a_3, a_4) \sim \Sigma(a_i),$$
$$\Sigma(a_i) = -mn\lambda_{m,n} + \sum_{i=1}^{3} \sum_{r,s}^{(m,n)} |\lambda_i - \lambda_{r,s}|_{\mathrm{Re}}, \tag{5}$$

the fusion set is $(m, n) = \{1 - m : 2 : m - 1, 1 - n : 2 : n - 1\}$.
The last expression was derived under the assumption that # of intermediate matter primary fields $= m \cdot n$.

We were able to modify and extend this formula to get rid of this restriction for the Lee-Yang series.

$$\Sigma_{MHEM} = \Sigma_{HEM} - \sum_{i=2}^{4} \sum_{(r,s)\in F_i \cap R_i} 2\lambda_{r,s} \ , \tag{6}$$

where F_i is the fusion set of $\Phi_{m,n}\Phi_i$ ($\Phi_{m,n}\Phi_i \to \Phi_{r,s}$) and R_i is the set of discrete terms in the OPE of $V_j V_k$ ($V_j V_k \to V_{r,-s}$) and $\{i, j, k\} = \{2, 3, 4\}$.

Idea of proof

From the definition we have

$$I_4(a_{m,-n}, a_2, a_3, a_4) = \sum_k \mathbb{C}_{1,2}^{M,k} \mathbb{C}_{k,3,4}^{M} \int' \mathrm{d}P \, \mathbb{C}_{1,2}^{L,P} \mathbb{C}_{P,3,4}^{L} \, \Phi(k, P), \tag{7}$$

where $\Phi(k, P)$ is an integral of conformal blocks over the moduli space.

If we take the limit $a_i \to$ degenerate values $a_{m_i,-n_i}$ then some of the structure constants $\mathbb{C}_{k,3,4}^{M}$ vanish in accordance with fusion rules. However the zeros may be annihilated by poles of $\Phi(k, P)$, and the old formula takes into account these extra terms. The latter can appear when there are appropriate discrete terms in the Liouville expression i.e. $\Delta_P^L + \Delta_k^M = 1$.

In the Liouville OPE the contour of integration is a real axis in the case when $\Re(|\lambda_i|) + \Re(|\lambda_j|) < Q/2$ for $i \neq j$. Otherwise it deforms and gets additional residue terms.

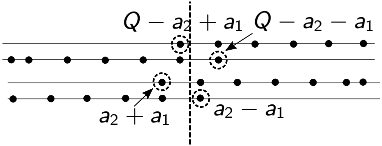

Figure: Poles of structure the constant and discrete terms.

In the framework of the Douglas equation approach there are two formulae for the four-point correlation numbers. First of them can be written as

$$\Sigma'_{DSE}(n_i) = -\hat{F}(0) + \sum_{i=1}^{4} \hat{F}(n_i)$$
$$-\hat{F}(\min(n_1+n_2, n_3+n_4)) - \hat{F}(\min(n_1+n_3, n_2+n_4)) - \hat{F}(\min(n_1+n_4, n_3+n_2)) \; , \tag{8}$$

where $\hat{F}(n) = (s+1-n)(s-n)\theta(n \leq s)$.
The second one coincides with the above one when the number of conformal blocks is maximal and does not otherwise.

The formula for four-point correlation numbers in Douglas equation approach is equivalent to the modified HEM formula:

$$\Sigma_{MHEM}(n_i) = \Sigma_{DSE}(n_i) \; .$$

Moreover, if there are no discrete terms in the operator product expansion $V_{1,n_2} V_{1,n_3}$, then we also have $\Sigma_{HEM} = \Sigma_{MHEM}$.

In HEM approach there was obtained a formula generalizing the old one. For Lee Yang series it coincides with the numerical computation and Douglas equation formula. Though this gives rise to lots of questions. For instance:

1. Is this approach also correct for other Minimal Models? For $\mathcal{M}(3/p)$, unitary series $\mathcal{M}(q, q+1)$ and others in Douglas equation approach it seems impossible to fulfill all the Minimal Model fusion rules.
2. Formula (6) was obtained as an analytic continuation to degenerate values of conformal dimensions/Liouville momenta. So it is natural to expect that multiplication of MM structure constants on delta-constants proportional to MM fusion rules will make some further corrections to this formula. It can probably bring some light on the impossibility to fullfil all fusion rules in the question above.

Thank you for your attention!