Relativistic bound state systems in light-front dynamics

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OUTLOOK

Aim: nonperturbative calculation of relativistic bound state properties in a well defined and controled approximation scheme

- Different forms of dynamics
- Properties of light-front dynamics
- Calculation of bound state properties
- Renormalization scheme
- First applications
- Role of antiparticle degrees of freedom
- Conclusions and perspectives

DIFFERENT FORMS OF DYNAMICS

❑ Dirac's classification

➢ Example: non-relativistic Schrödinger equation for a two-body system

- $H\ \Psi = i\dfrac{\partial \Psi}{\partial t}$ for a stationary state $\Psi(\vec{r},t) = \phi(\vec{r})e^{-iEt}$

  with $H\ \phi = E\ \phi$

- the wave function is defined on the plane t = 0

- the evolution of the wave function is driven by the Hamiltonian

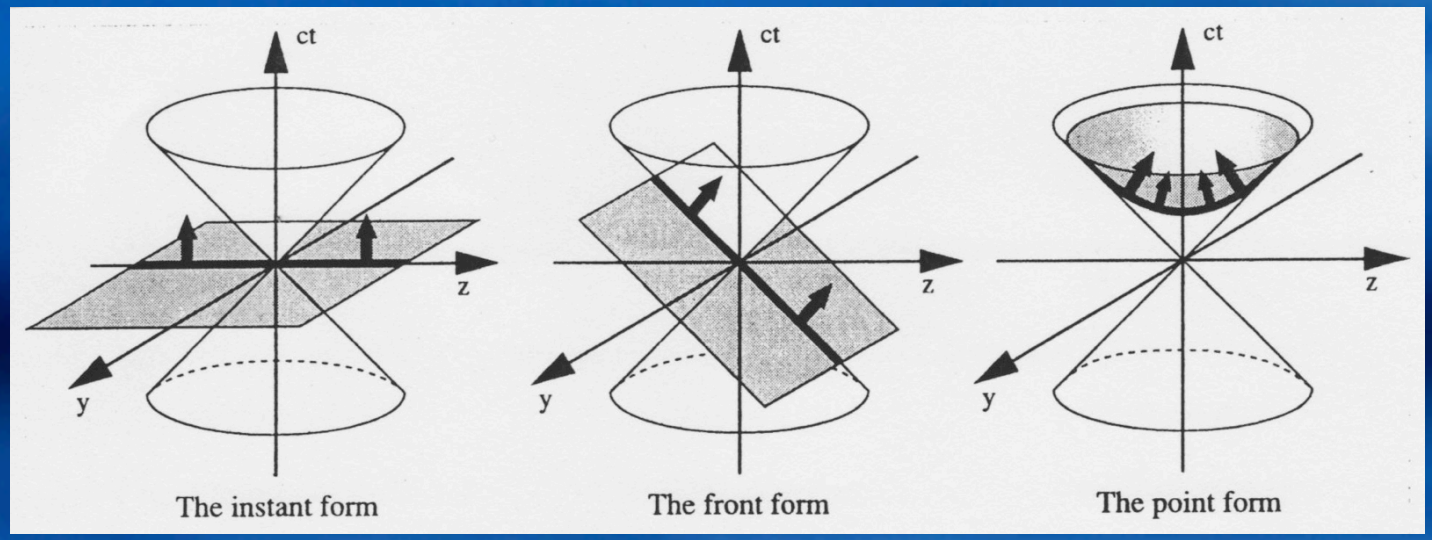
  ➥ dynamical operator : it depends upon the interaction

- the rotations of the system leave the plane t = 0 invariant

  ➥ kinematical operator Y_{lm} :

  it does not depend upon the interaction

➢ Dirac's classification: there exists three different forms of dynamics depending upon the plane on which the state vector is defined



for 2 dimensions in space and 1 in time

$$t = cte \qquad \tau = ct + z = cte \qquad \tau = \sqrt{c^2t^2 - \vec{x}^2} = cte$$

Rem: in the non-relativistic limit, $c \longrightarrow \infty$ all forms of dynamics are equivalent to the instant form.

❑ Generators of the Poincaré group

➢ **Poincaré group: relativistic extension of the Galilean group, which governs the physical transformations which leave the state vector invariant**

P^0, P^i	• **translations** in time (1) and space (3)	4 generators
M^{ij}	• **rotations** in space (3)	3 generators
$M^{0i} = K^i$	• Lorentz **boosts**	3 generators

➢ Properties of these generators

Equal time $t = 0$

- 3 translations in space are kinematical
- 3 rotations are kinematical

- 1 translation in time is dynamical
- 3 boosts are dynamical

4 dynamical generators

6 kinematical generators

➥ Convenient for non-relativistic systems

Light front time $t^+ = t + z = 0$

- 3 translations in space are kinematical
- 1 rotation is kinematical
- 3 boosts are kinematical

- 1 translation in LF time is dynamical
- 2 rotations are dynamical

3 dynamical generators

7 kinematical generators

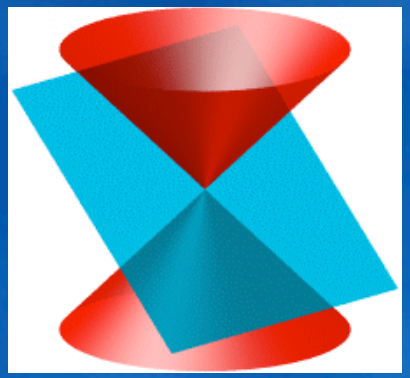
among it: boost along the z direction

➥ interest in describing physical systems boosted to high energy

➥ Convenient for relativistic systems

PROPERTIES OF LIGHT FRONT DYNAMICS

❑ Vacuum structure



- from $P^2 \geq 0$ and $P^0 \geq 0$

  one gets $P^+ \geq 0$ $P^- \geq 0$

  with $P^+ = E + P_z$ and $P^- = E - P_z$

- the vacuum is defined by $P^- = P^+ = 0$

  free vacuum

- it is non degenerate unlike in instant form where you can have particle states with total momentum zero with particle momenta +k and -k

- the nonperturbative properties of the (standard) vacuum are now embedded in zero-mode contributions ($k^+=0$) to the field operator

  ("vacuum sector")

T. Heinzl et al, Z. Phys. C 56 (1992) 415

❑ Fock state decomposition

➢ Decomposition of the state vector in Fock components
with a definite number of particles

$$|\Phi\rangle = |1\rangle + |2\rangle + \ldots$$

- many-body system reminiscent of non-relativistic nuclear physics
- many-body wave function : coefficient of each Fock component
- the Fock components are invariant under boosts: there is no mixing between Fock components by boost

➢ One has however to be carefull because the Lagrangian is singular:
contact interactions, end point singularities, zero modes ...

➢ One important drawback:

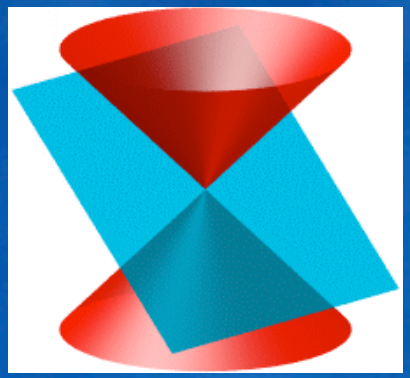
➥ the rotations in the transverse direction are DYNAMICAL

❑ Rotational invariance

➢ The choice of the plane $t + z = 0$ violates rotational symmetry

➢ One way out: keep the position of the Light Front plane arbitrary $\vec{n}$

$$\omega.x = \omega_0 t - \vec{\omega}\vec{x} = 0 \qquad \omega^2 = 0 \qquad \omega = (1, \vec{n})$$



- Explicitly covariant formulation of light-front dynamics

V. Karmanov 1976, J. Carbonell et al, Phys. Rep. 300 (1998) 215

- new degree of freedom ω or $\vec{n}$
- if $\omega = (1, 0, 0, -1)$ one recovers the standard formulation

➢ Any physical observable should not depend on ω in any exact calculation, but does if approximations are made. The violation is explicit

➢ These nonphysical ω -dependent contributions can serve as a quantitative estimate of the approximation

➢ Wave functions do depend on ω

❑ Angular condition

➢ **Calculation of the angular momentum operator on the light-front**

$$\hat{J}_{\mu\nu} = \hat{J}^0_{\mu\nu} + \hat{J}^{int}_{\mu\nu}$$

where $\hat{J}^{int}_{\mu\nu}$ **is a dynamical operator**

$$\hat{J}^{int}_{\mu\nu} = \int H^{int}(x)(x_\mu \omega_\nu - x_\nu \omega_\mu)\delta(\omega \cdot x - \sigma)\ d^4x$$

➢ **Using the Poincaré group equations for the state vector** ϕ

$$\hat{J}^{int}_{\mu\nu}\phi = i\left(\omega_\mu \frac{\partial}{\partial \omega_\nu} - \omega_\nu \frac{\partial}{\partial \omega_\mu}\right)\phi$$

➥ **simple (kinematical) construction of the angular momentum operator in terms of the (dynamical) dependence of the state vector on the position of the light-front**

CALCULATION OF BOUND STATE PROPERTIES

- **Eigenvalue equation**

$$\hat{P}^2\,\phi(p) = M^2\,\phi(p) \quad \text{with} \quad \hat{P}_\mu = \hat{P}^0_\mu + \hat{P}^{int}_\mu$$

$$\hat{P}^{int}_\mu = \omega_\mu \int H^{int}(x)\delta(\omega{\cdot}x)\,d^4x = \omega_\mu \int_{-\infty}^{+\infty} \tilde{H}^{int}(\omega\tau)\frac{d\tau}{2\pi}$$

➢ **Decomposition of the state vector**

Off-shell energy

$$\phi(p) = \sum_{n=1}^{\infty} \int dD_n \phi_n(k_1,\ldots,k_n;p)\delta^4(k_1+\ldots+k_n-p-\omega\tau_n\,|n\rangle$$

with $2\,\omega{\cdot}p\,\tau_n = (s_n - M^2)$ $\qquad s_n = (k_1+\ldots+k_n)^2$

and the normalization $\phi(p')^\dagger\phi(p) = 2p_0\;\delta^{(3)}(\mathbf{p}'-\mathbf{p})$

➢ Final eigenvalue equation

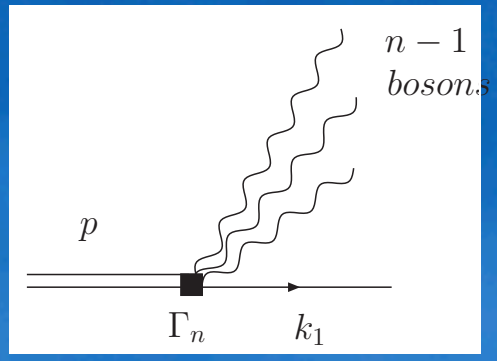
with the momentum conservation law for each Fock component

one can rewrite the equation of motion $\hat{P}^2\,\phi(p) = M^2\,\phi(p)$ as

$$\frac{1}{2\pi}\int \tilde{H}^{int}(\omega\tau)\frac{d\tau}{\tau}\mathcal{G}(p) = -\mathcal{G}(p)$$

with $$\mathcal{G}(p) = 2(\omega\cdot p)\hat{\tau}\phi(p)$$

and $$\hat{\tau}\phi_n = \tau_n\phi_n \equiv \bar{u}(k_1)\Gamma_n u(p)$$



for a one constituent fermion coupled to bosons

➢ System of coupled integral equations for the vertex functions Γ_n

❑ Spin decomposition of the Fock components

➢ **The two-body vertex function depends on two kinematical scalar variables**

$$\Gamma_2(x, \mathbf{R}_\perp^2) \quad \textbf{with} \quad x = \frac{\omega.k_2}{\omega.p} \quad \textbf{and} \quad R = k_2 - x\, p = (R_0, \mathbf{R}_\perp, R_0)$$

➢ **Spin decomposition (in the Yukawa model)**

$$\bar{u}(k_1)\Gamma_2 u(p) = \bar{u}(k_1)\left(b_1 + b_2 \frac{m\ \not{\omega}}{\omega.p}\right) u(p)$$

Rem: four independent spin components for the three-body vertex function (and higher orders as well)

❑ Physical observables

➢ Electromagnetic form factor of a spin 1/2 fermion

$$\Gamma_\rho = F_1\gamma_\rho + \frac{iF_2}{2M}\sigma_{\rho\nu}q^\nu$$

$$+B_1\left(\frac{\not\omega}{\omega\cdot p} - \frac{1}{(1+\eta)M}\right)P_\rho + B_2\frac{M}{\omega\cdot p}\omega_\rho + B_3\frac{M^2}{(\omega\cdot p)^2}\not\omega\omega_\rho$$

with $q = p' - p$ and $P = p' + p$

➢ **Two physical** form factors $F_{1,2}$

➢ **Three unphysical** form factors $B_{1,2,3}$

- they should be zero in an exact calculation
- should be extracted out to get the physical form factors in any approximate calculation

RENORMALIZATION SCHEME

One constituent fermion coupled to (N-1) bosons
without vacuum polarization of the bosons in a first approximation
(light-front quenched approximation)

❑ Mass counterterm

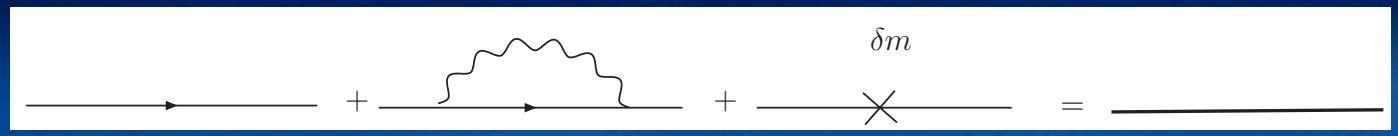
- State vector $\phi(p)$ projected on a basis of free physical states with physical masses for the constituent fermion (m) and bosons (μ)
  - ➥ most natural choice rather than an expansion on a basis of bare particles with (infinite) bare masses
- Solution of the eigenstate equation $P^2\phi(p) = M^2\phi(p)$
- Determination of the mass counterterm δm in the limit $m \to M$

Rem. Regularization using Pauli-Villars auxiliary fields

❑ Control of uncancelled divergences

➢ cancellation of divergences involve different Fock sectors

➥ uncancelled divergences when the Fock space is truncated to order N



➢ δm , g_0 should thus translate into a sequence $\delta m^{(n)}$, $g_0^{(n)}$

where n is the number of constituent in flight

Rem: reminiscent of perturbation theory when δm , $g_0 \to \delta m^{(n)}$, $g_0^{(n)}$

where here n is the order of the perturbative expansion

➢ Fock sector dependent renormalization scheme to be checked in explicit calculations

V. Karmanov et al, Phys. Rev. D77 (2008) 085028

Physical observables should be regularization-scale invariant order by order in the Fock expansion

V. Karmanov et al, Phys. Rev. D82 (2010) 056010

➢ Renormalization condition

$$\Gamma_2^{(N)}(s_2 = M^2) = g\sqrt{I_1^{(N-1)}}$$

weight of the one-body component (single constituent fermion) in the normalized physical state, calculated in the N-1 Fock space truncation

this implies

$$b_1^{(N)}(s_2 = M^2) = g\sqrt{I_1^{(N-1)}}$$

$$b_2^{(N)}(s_2 = M^2) = 0$$

with

$$\bar{u}(k_1)\Gamma_2^{(N)}u(p) = \bar{u}(k_1)\left[b_1^{(N)} + b_2^{(N)}\frac{m\ \not{\omega}}{\omega\cdot p}\right]u(p)$$

Rem: only possible if we keep the ω dependence explicitly!

➥ **First condition** $b_1^{(N)}(s_2 = M^2) = g\sqrt{I_1^{(N-1)}}$

- implies that $b_1^{(N)}(s_2 = M^2)$ should be independent of x at $\mathbf{R}_\perp^2$ fixed by the condition $s_2 = M^2$

- if not, this condition determines uniquely an x-dependent bare coupling constant $g_0^{(N)}(x)$

- this dependence on x can serve as an estimate of the accuracy of the calculation

➥ **Second condition** $b_2^{(N)}(s_2 = M^2) = 0$

- implies that $b_2^{(N)}(s_2 = M^2)$ should be zero and independent of x at $\mathrm{R}_\perp^2$ fixed by the condition $s_2 = M^2$

- these conditions should be enforced with an appropriate ω - dependent counterterm $Z_\omega^{(N)}(x)$ if necessary, fixed also uniquely by this condition

- the size of this counterterm can serve as an estimate of the accuracy of the calculation

How this works in practice!

➢ $\delta m^{(n)}$ and $g_0^{(n)}$ calculated by successive solutions of the N=1, N=2 , N=3 ...N systems

- N=1 $\delta m^{(1)} = 0$
- N=2 $\delta m^{(2)}$ determined to get $\hat{P}^2\,\phi(p) = M^2\,\phi(p)$

$$\delta m^{(2)} = -\Sigma(p^2 = M^2)$$

$g_0^{(2)}$ determined to get

$$\bar{u}(k_1)\Gamma_2(s = M^2)u(p) = g_{phys}\bar{u}(k_1)u(p)$$

- etc...

➢ **This is a systematic, nonperturbative, procedure which should avoid uncancelled divergences**

but also true if there are no divergences!

FIRST APPLICATIONS

Regularization with one Pauli-Villars fermion (m_1) and one Pauli-Villars boson (μ_1)

❑ QED in the two-body Fock space truncation

➢ should be identical to the standard result in second order of perturbation theory without any expansion in g !

➢ this is achieved only if the nonperturbative renormalization conditions are correctly formulated

➢ in that case one finds

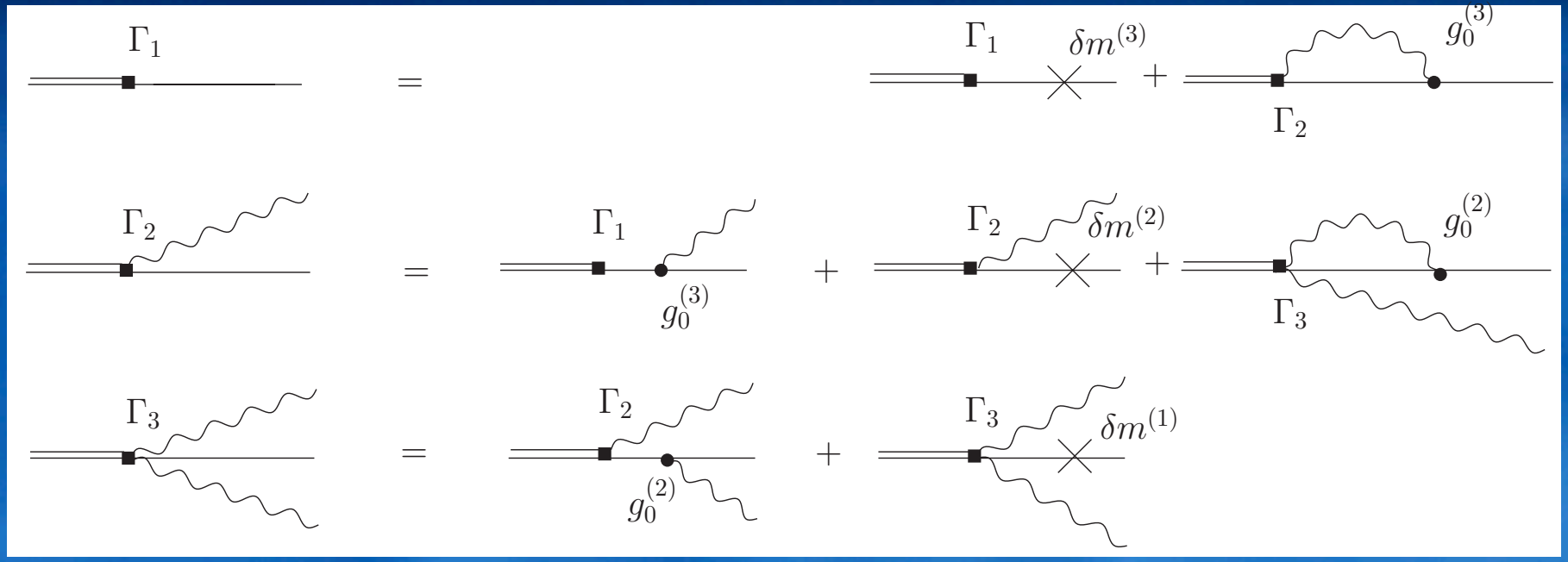
- $e_0 \equiv e$ to all orders in the Fock expansion
- anomalous magnetic moment $\dfrac{\alpha}{2\pi}$ (Schwinger correction)

➢ at that order we have $b_1^{(2)} = cte \qquad b_2^{(2)} = 0$

❑ Yukawa model in the three-body Fock space truncation

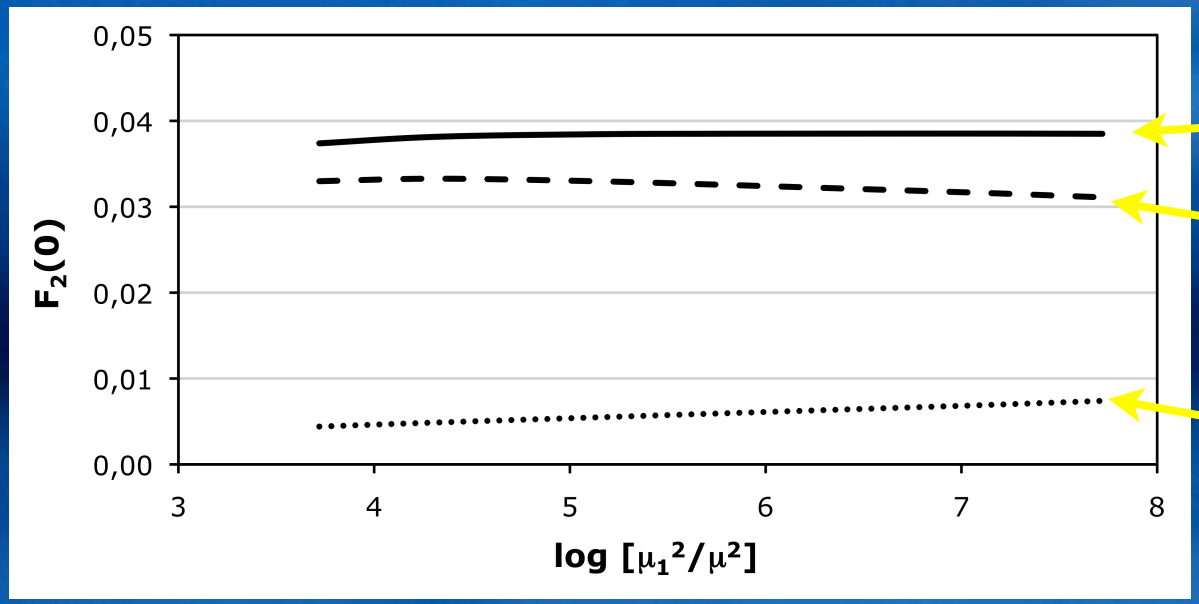
➢ first non trivial nonperturbative calculation

➥ resummation of overlapping divergences to all orders



- calculation for M = 0.938 GeV and μ_1 = 0.138 GeV (scalar pion)
- analytical limit to infinity taken for the Pauli-Villars fermion mass
- relative numerical accuracy of the order of 5 10^{-3} for μ_1 = 10 - 1000 GeV
- calculation done on a laptop (few hours)

anomalous magnetic moment of the physical fermion

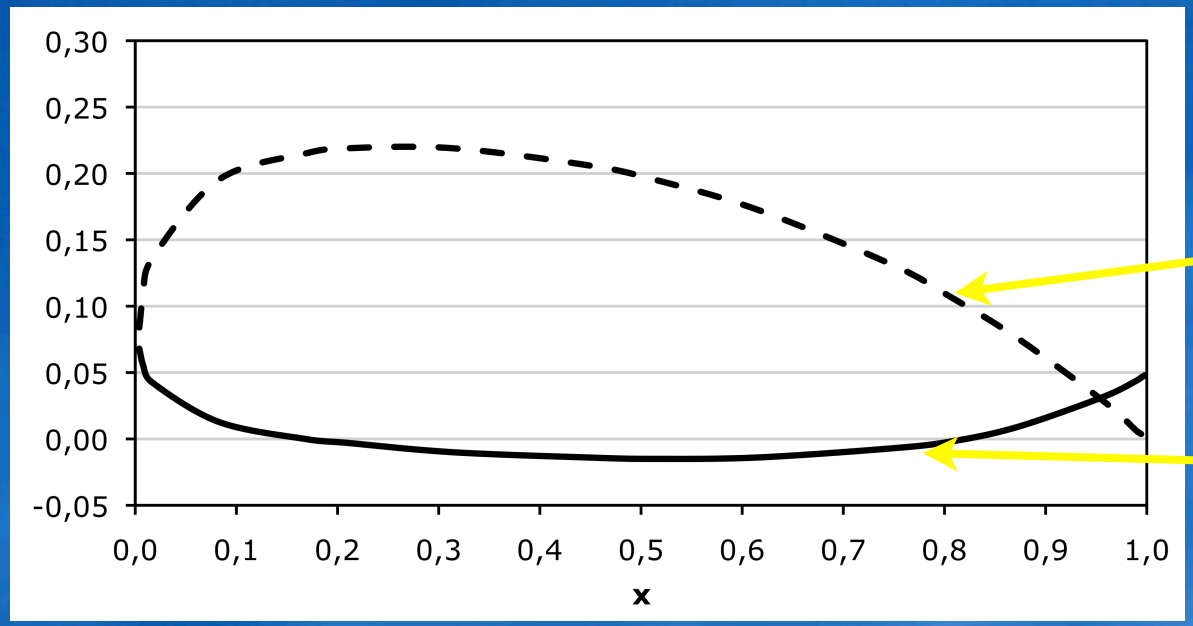


total

two-body

three-body

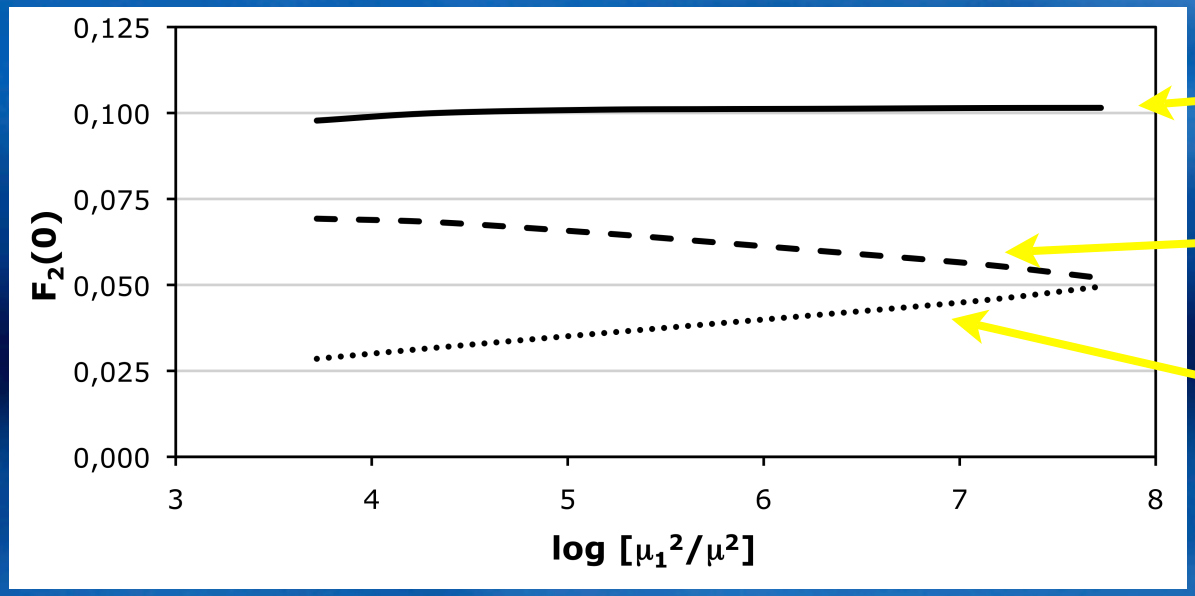
$\alpha = 0.2$



$$\frac{2(1-x)Z_\omega(x)}{(2-x)\bar{g}_0}$$

$$\frac{\delta g_0(x)}{\bar{g}_0}$$

➥ correction to the calculation in the two-body Fock space truncation: 3 %

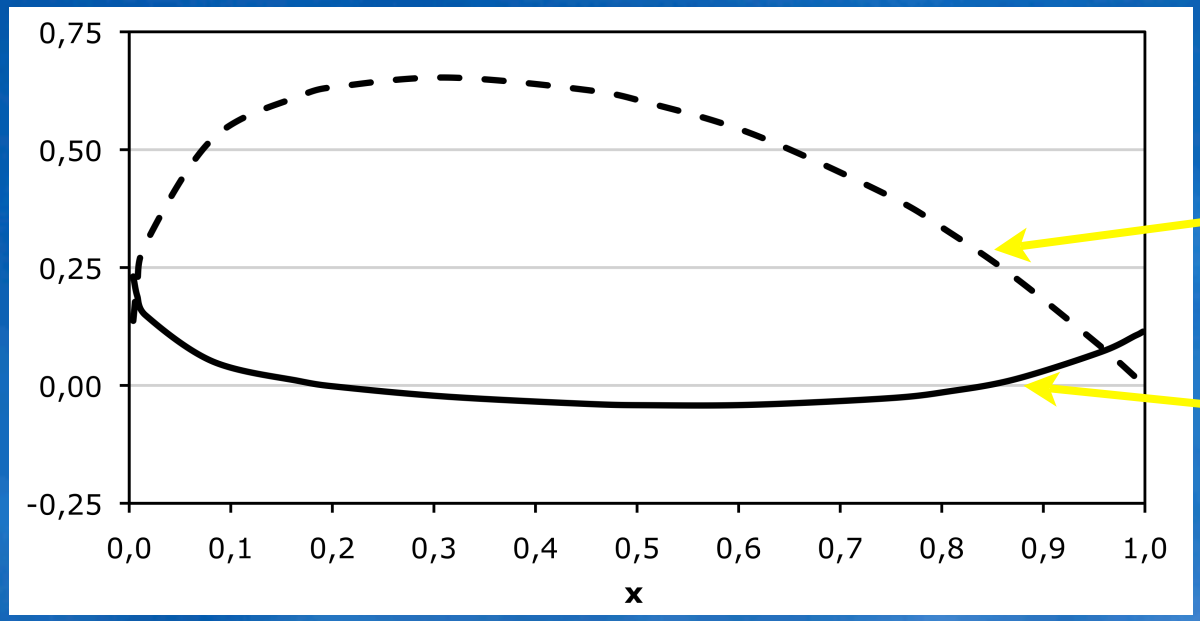


total

two-body

three-body

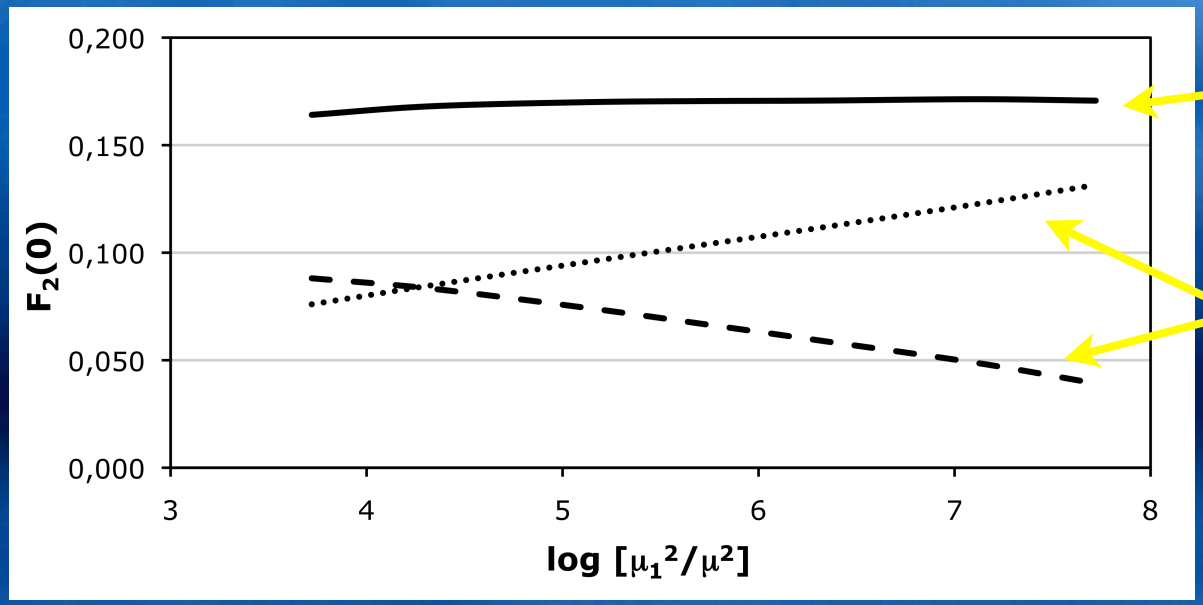
$\alpha = 0.5$



$$\frac{2(1-x)Z_\omega(x)}{(2-x)\bar{g}_0}$$

$$\frac{\delta g_0(x)}{\bar{g}_0}$$

➥ correction to the calculation in the two-body Fock space truncation: 9 %

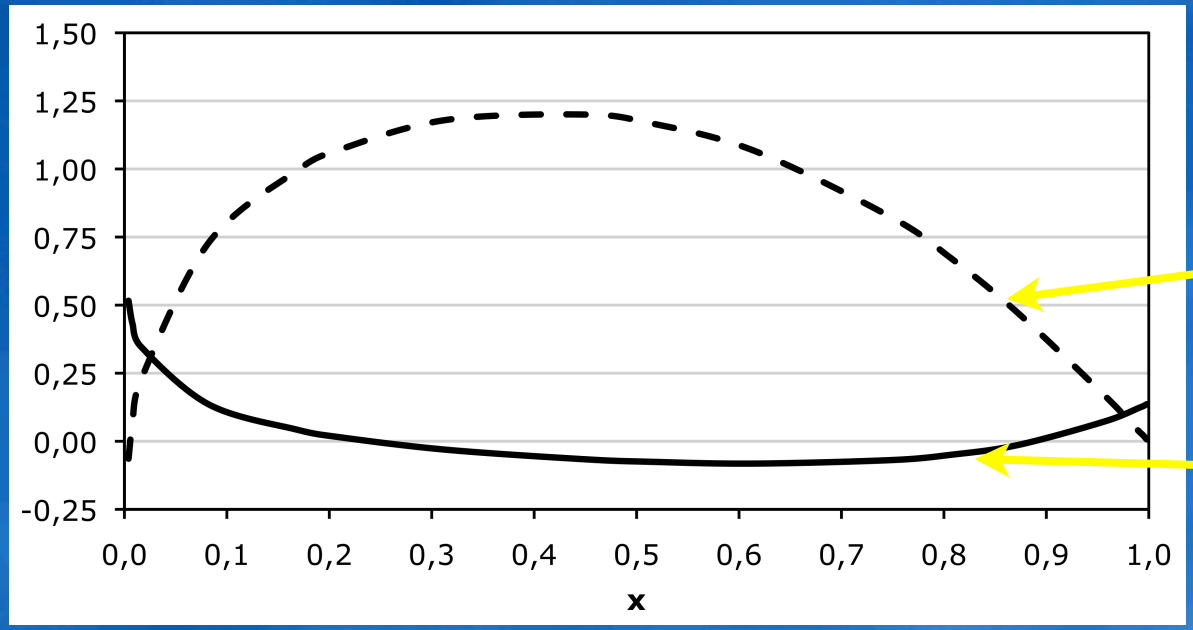


total

two-body

three-body

$\alpha = 0.8$



$$\frac{2(1-x)Z_\omega(x)}{(2-x)\bar{g}_0}$$

$$\frac{\delta g_0(x)}{\bar{g}_0}$$

➥ correction to the calculation in the two-body Fock space truncation: 15 %

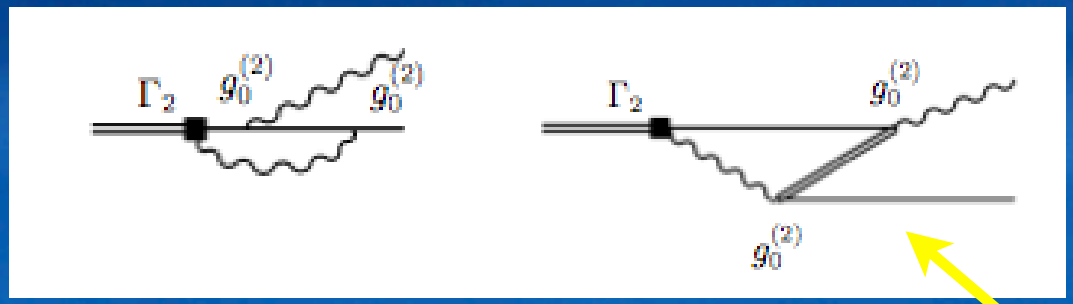
➥ regularization-scale invariance realized in our framework with an accuracy better then 5 10^{-3} even for $\alpha = 1$

➥ this regularization-scale invariance does allow to make predictions for physical observables at each order of the Fock expansion and test the convergence of the Fock expansion

➥ one needs however to investigate the origin of the non-zero ω - dependent counterterm and the x-dependence of the bare coupling constant!

ROLE OF ANTIPARTICLE DEGREES OF FREEDOM

In the light-front quenched approximation, still contributions from antiparticles

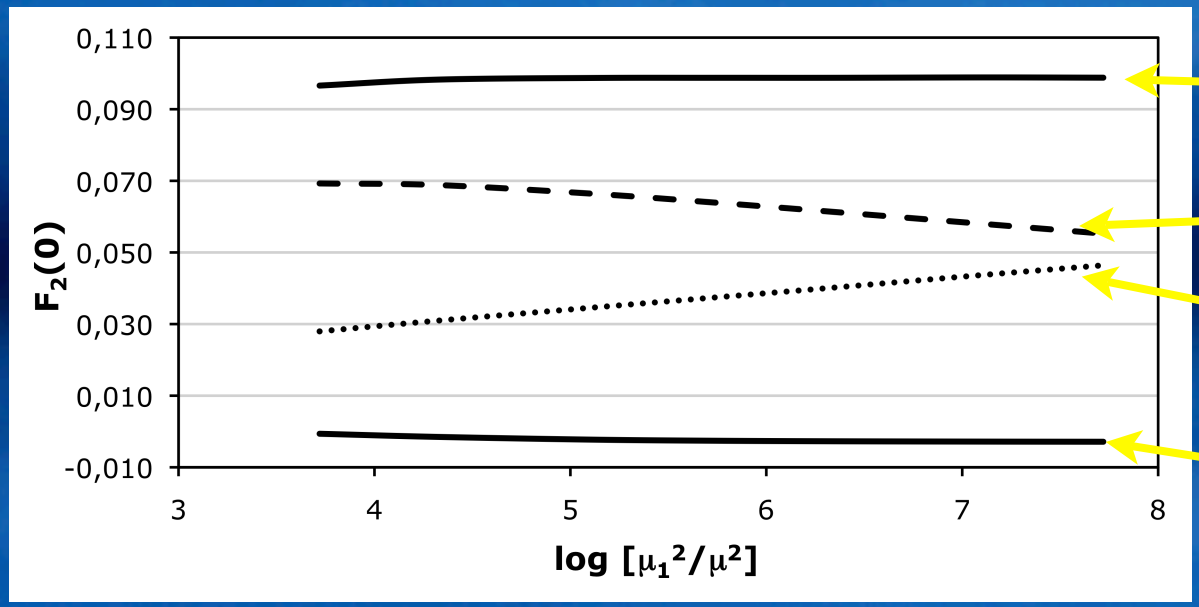


for the N=3 calculation

irreducible first order contribution to consider in the eigenvalue equation

➢ **Necessary in order to get full equivalence with Feynman perturbation theory (resummed to all orders)**

❑ Anomalous magnetic moment



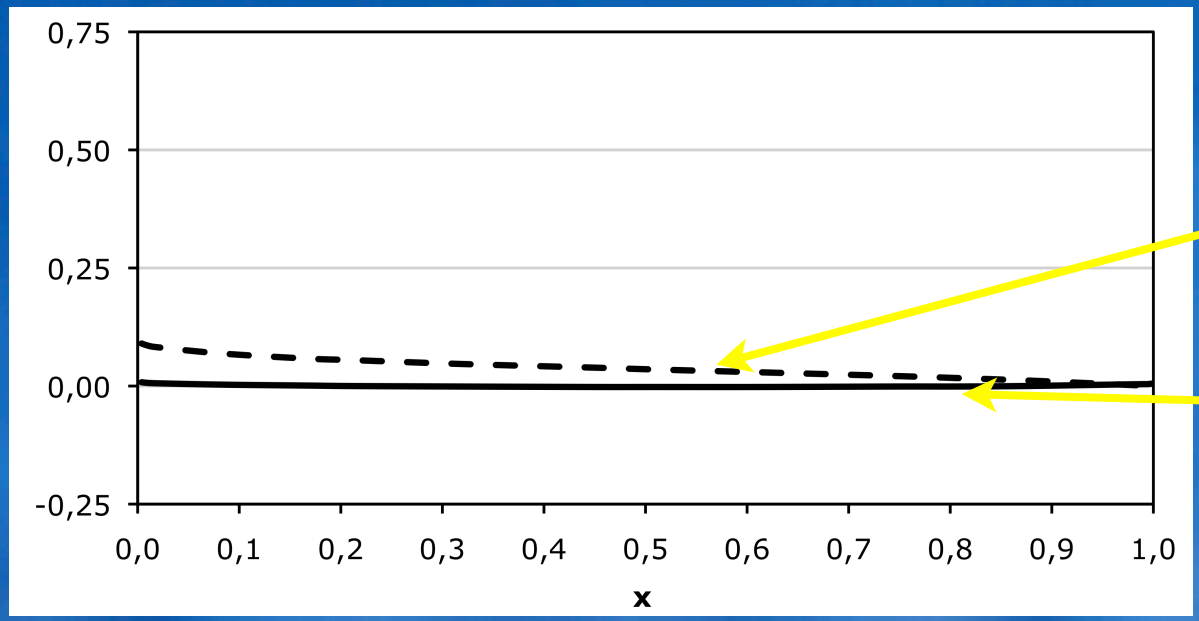
total

two-body

three-body

anti-fermion

$\alpha = 0.5$



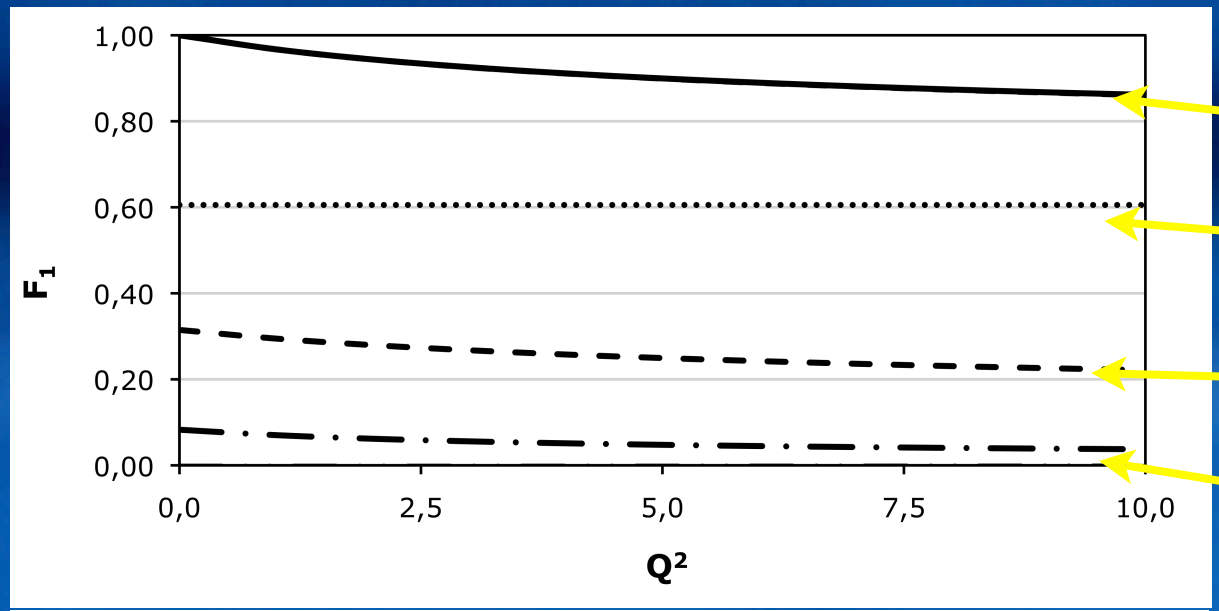
$$\frac{2(1-x)Z_\omega(x)}{(2-x)\bar{g}_0}$$

$$\frac{\delta g_0(x)}{\bar{g}_0}$$

➥ but slight regularization-scale non invariance for higher α : this first order contribution does not exhaust all contributions necessary to restore equivalence with Feynman perturbation theory

❑ Electromagnetic form factor $F_1(Q^2)$

$\mu_1 = 50\ GeV$



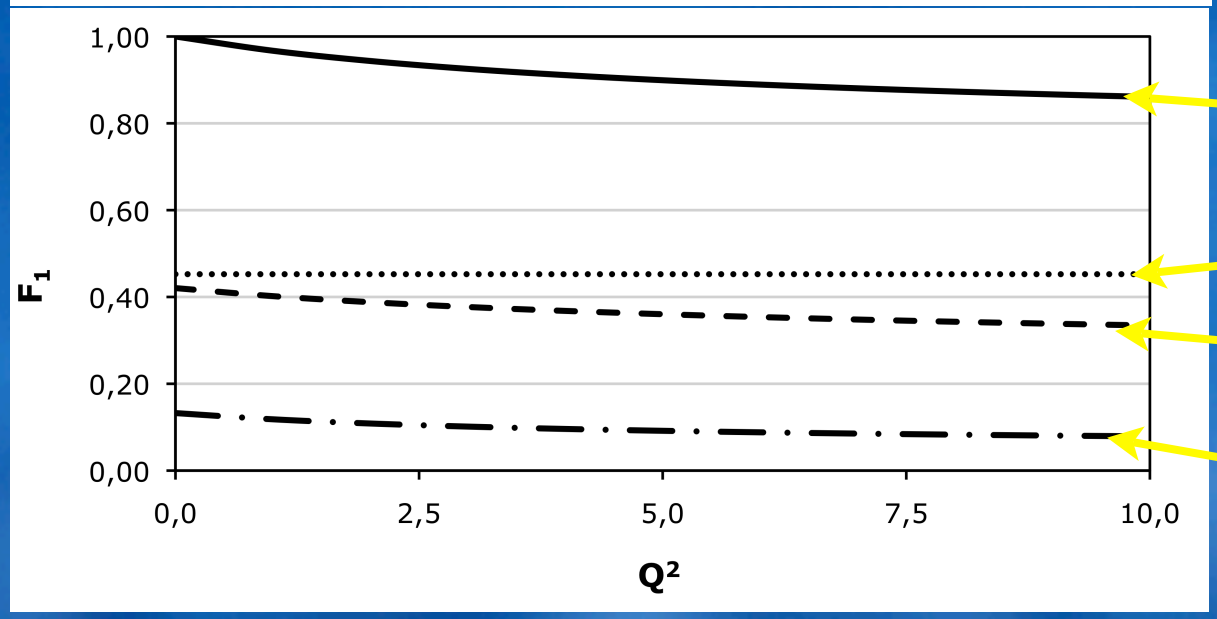
total

one-body

two-body

three-body

$\alpha = 0.5$



total

one-body

two-body

three-body

$\mu_1 = 200\ GeV$

CONCLUSIONS ....

- **Well defined and controled nonperturbative framework**
  - ➢ Regularization scale invariance order by order in the Fock expansion
  - ➢ Check with results in perturbation theory (resummed to all orders) in the two-body Fock space truncation
  - ➢ Non trivial nonperturbative calculations in the three-body Fock space truncation
  - ➢ Calculation of physical observables

... AND PERSPECTIVES

- **Systematic regularization scheme with no infinities at all, and no non-physical particles like Pauli-Villars particles**

  - Taylor-Lagrange regularization scheme based on the properties of fields as distributions acting on test functions with specific mathematical properties

    P. Grangé and E. Werner P. Grangé et al. Phys. Rev. D80 (2009) 105012

- **Application to chiral perturbation theory**

  - Light front chiral effective field theory based on the Fock expansion of the state vector

- **Systems with spontaneous symmetry breaking in a Fock expansion on the light-front**

  - Zero modes in 3+1 dimensions in the $\lambda\ \Phi^4$ model

Thank you for your attention !