Relativistic bound state systems in light-front dynamics

Jean-François Mathiot
Laboratoire de Physique Corpusculaire
Clermont-Ferrand, France

In collaboration with

V. Karmanov (Moscow), A. Smirnov (Moscow) and N. Tsirova (Samara)
Aim: nonperturbative calculation of relativistic bound state properties in a well defined and controlled approximation scheme

- Different forms of dynamics
- Properties of light-front dynamics
- Calculation of bound state properties
- Renormalization scheme
- First applications
- Role of antiparticle degrees of freedom
- Conclusions and perspectives
DIFFERENT FORMS OF DYNAMICS

 Dirac’s classification

 ➤ Example: non-relativistic Schrödinger equation for a two-body system

\[ H \Psi = i \frac{\partial \Psi}{\partial t} \]

for a stationary state \( \Psi(\vec{r}, t) = \phi(\vec{r}) e^{-iEt} \)

with \( H \phi = E \phi \)

• the wave function is defined on the plane \( t = 0 \)

• the evolution of the wave function is driven by the Hamiltonian

 ➤ dynamical operator: it depends upon the interaction

• the rotations of the system leave the plane \( t = 0 \) invariant

 ➤ kinematical operator \( Y_{lm} \):

it does not depend upon the interaction

➤ Dirac’s classification: there exists three different forms of dynamics depending upon the plane on which the state vector is defined
Rem: in the non-relativistic limit, $c \to \infty$ all forms of dynamics are equivalent to the instant form.

Generators of the Poincaré group

- **Poincaré group**: relativistic extension of the Galilean group, which governs the physical transformations which leave the state vector invariant

- $P^0, P^i$ • translations in time (1) and space (3)  
  4 generators

- $M^{ij}$ • rotations in space (3)  
  3 generators

- $M^{0i} = K^i$ • Lorentz boosts  
  3 generators
Properties of these generators

**Equal time** \( t = 0 \)
- 3 translations in space are kinematical
- 3 rotations are kinematical
- 1 translation in time is dynamical
- 3 boosts are dynamical

4 dynamical generators
6 kinematical generators

**Light front time** \( t^+ = t + z = 0 \)
- 3 translations in space are kinematical
- 1 rotation is kinematical
- 3 boosts are kinematical
- 1 translation in LF time is dynamical
- 2 rotations are dynamical

3 dynamical generators
7 kinematical generators

Among it: boost along the z direction

➤ interest in describing physical systems boosted to high energy

➤ Convenient for non-relativistic systems

➤ Convenient for relativistic systems
PROPERTIES OF LIGHT
FRONT DYNAMICS

❑ Vacuum structure

- from $P^2 \geq 0$ and $P^0 \geq 0$
- one gets $P^+ \geq 0$ and $P^- \geq 0$
- with $P^+ = E + P_z$ and $P^- = E - P_z$

- the vacuum is defined by free vacuum $P^- = P^+ = 0$

- it is non degenerate unlike in instant form where you can have particle states with total momentum zero with particle momenta $+k$ and $-k$

- the nonperturbative properties of the (standard) vacuum are now embedded in zero-mode contributions ($k^+=0$) to the field operator ("vacuum sector")

Fock state decomposition

> Decomposition of the state vector in **Fock components** with a **definite number of particles**

\[ |\Phi\rangle = |1\rangle + |2\rangle + \ldots \]

- many-body system reminiscent of non-relativistic nuclear physics
- **many-body wave function**: coefficient of each Fock component
- the Fock components are **invariant under boosts**: there is no mixing between Fock components by boost

> One has however to be careful because **the Lagrangian is singular**: contact interactions, end point singularities, zero modes ...

> One **important drawback**:

⇒ the rotations in the transverse direction are **DYNAMICAL**
Rotational invariance

➢ The choice of the plane \( t + z = 0 \) violates rotational symmetry

➢ One way out: keep the position of the Light Front plane \textbf{arbitrary} \( \vec{n} \)

\[
\omega \cdot x = \omega_0 t - \vec{\omega} \vec{x} = 0 \quad \omega^2 = 0 \quad \omega = (1, \vec{n})
\]

• Explicitly covariant formulation of light-front dynamics

• new degree of freedom \( \omega \) or \( \vec{n} \)

• if \( \omega = (1, 0, 0, -1) \) one recovers the standard formulation

➢ Any physical observable should not depend on \( \omega \) in any exact calculation, but does if approximations are made. The violation is explicit

➢ These nonphysical \( \omega \)-dependent contributions can serve as a quantitative estimate of the approximation

➢ Wave functions do depend on \( \omega \)
Angular condition

Calculation of the angular momentum operator on the light-front

\[ \hat{J}_{\mu \nu} = \hat{J}_{\mu \nu}^0 + \hat{J}_{\mu \nu}^{int} \]

where \( \hat{J}_{\mu \nu}^{int} \) is a dynamical operator

\[ \hat{J}_{\mu \nu}^{int} = \int H^{int}(x)(x_\mu \omega_\nu - x_\nu \omega_\mu) \delta(\omega \cdot x - \sigma) \, d^4x \]

Using the Poincaré group equations for the state vector \( \phi \)

\[ \hat{J}_{\mu \nu}^{int} \phi = i \left( \omega_\mu \frac{\partial}{\partial \omega_\nu} - \omega_\nu \frac{\partial}{\partial \omega_\mu} \right) \phi \]

simple (kinematical) construction of the angular momentum operator in terms of the (dynamical) dependence of the state vector on the position of the light-front
CALCULATION OF BOUND STATE PROPERTIES

- **Eigenvalue equation**

\[
\hat{P}^2 \phi(p) = M^2 \phi(p) \quad \text{with} \quad \hat{P}_\mu = \hat{P}_\mu^0 + \hat{P}_\mu^{\text{int}}
\]

\[
\hat{P}_\mu^{\text{int}} = \omega_\mu \int H^{\text{int}}(x) \delta(\omega \cdot x) \, d^4x = \omega_\mu \int H^{\text{int}}(\omega \tau) \frac{d \tau}{2\pi}
\]

- **Decomposition of the state vector**

\[
\phi(p) = \sum_{n=1}^{\infty} \int dD_n \phi_n(k_1, \ldots, k_n; p) \delta^4(k_1 + \ldots + k_n - p - \omega \tau_n | n)
\]

with \(2 \omega \cdot p \tau_n = (s_n - M^2)\)

\[
s_n = (k_1 + \ldots + k_n)^2
\]

and the normalization \(\phi(p') \dagger \phi(p) = 2p_0 \delta^{(3)}(p' - p)\)

**Off-shell energy**

\[
\phi(p') \dagger \phi(p) = 2p_0 \delta^{(3)}(p' - p)
\]
Final eigenvalue equation

with the momentum conservation law for each Fock component one can rewrite the equation of motion $\hat{P}^2 \phi(p) = M^2 \phi(p)$ as

$$\frac{1}{2\pi} \int \tilde{H}^{int}(\omega \tau) \frac{d\tau}{\tau} G(p) = -G(p)$$

with

$$G(p) = 2(\omega \cdot p) \hat{\tau} \phi(p)$$

and

$$\hat{\tau} \phi_n = \tau_n \phi_n \equiv \tilde{u}(k_1) \Gamma_n u(p)$$

for a one constituent fermion coupled to bosons

System of coupled integral equations for the vertex functions $\Gamma_n$
Spin decomposition of the Fock components

The two-body vertex function depends on two kinematical scalar variables

\[ \Gamma_2(x, R^2_\perp) \quad \text{with} \quad x = \frac{\omega \cdot k_2}{\omega \cdot p} \quad \text{and} \quad R = k_2 - x \ p = (R_0, R_\perp, R_0) \]

Spin decomposition (in the Yukawa model)

\[ \bar{u}(k_1) \Gamma_2 u(p) = \bar{u}(k_1) \left( b_1 + b_2 \frac{m \ \omega}{\omega \cdot p} \right) u(p) \]

Rem: four independent spin components for the three-body vertex function (and higher orders as well)
Physical observables

Electromagnetic form factor of a spin 1/2 fermion

\[ \Gamma_\rho = F_1 \gamma_\rho + \frac{\imath F_2}{2M} \sigma_{\rho\nu} q^\nu \]

\[ + B_1 \left( \frac{\omega}{\omega \cdot p} - \frac{1}{(1 + \eta)M} \right) P_\rho + B_2 \frac{M}{\omega \cdot p} \omega_\rho + B_3 \frac{M^2}{(\omega \cdot p)^2} \psi \omega_\rho \]

with \( q = p' - p \) and \( P = p' + p \)

Two physical form factors \( F_{1,2} \)

Three unphysical form factors \( B_{1,2,3} \)

- they should be zero in an exact calculation
- should be extracted out to get the physical form factors in any approximate calculation
One constituent fermion coupled to (N-1) bosons without vacuum polarization of the bosons in a first approximation (light-front quenched approximation)

- **Mass counterterm**

  - State vector \( \phi(p) \) projected on a basis of free physical states with physical masses for the constituent fermion \( m \) and bosons \( \mu \)

  - Most natural choice rather than an expansion on a basis of bare particles with (infinite) bare masses

  - Solution of the eigenstate equation

\[ P^2 \phi(p) = M^2 \phi(p) \]

- Determination of the mass counterterm \( \delta m \) in the limit \( m \to M \)

Rem. Regularization using Pauli-Villars auxiliary fields
Control of uncancelled divergences

- Cancellation of divergences involve different Fock sectors
- Uncancelled divergences when the Fock space is truncated to order N

\[ \mathcal{F} \cdot \mathcal{G} + \delta m = \delta m^{(n)} + g_0^{(n)} \]

\[ \delta m, g_0 \text{ should thus translate into a sequence } \delta m^{(n)}, g_0^{(n)} \]

Where \( n \) is the number of constituent in flight

Rem: reminiscent of perturbation theory when \( \delta m, g_0 \to \delta m^{(n)}, g_0^{(n)} \)

Where here \( n \) is the order of the perturbative expansion

- Fock sector dependent renormalization scheme to be checked in explicit calculations


Physical observables should be regularization-scale invariant order by order in the Fock expansion
Renormalization condition

\[ \Gamma_2^{(N)}(s_2 = M^2) = g \sqrt{I_1^{(N-1)}} \]

this implies

\[ b_1^{(N)}(s_2 = M^2) = g \sqrt{I_1^{(N-1)}} \]

\[ b_2^{(N)}(s_2 = M^2) = 0 \]

with

\[ \bar{u}(k_1) \Gamma_2^{(N)} u(p) = \bar{u}(k_1) \left[ b_1^{(N)} + b_2^{(N)} \frac{m \not{\omega}}{\omega \cdot p} \right] u(p) \]

Rem: only possible if we keep the \( \omega \) dependence explicitly!
First condition: \[ b_{1}^{(N)}(s_{2} = M^{2}) = g \sqrt{I_{1}^{(N-1)}} \]

- implies that \( b_{1}^{(N)}(s_{2} = M^{2}) \) should be independent of \( x \) at \( R_{1}^{2} \) fixed by the condition \( s_{2} = M^{2} \)

- if not, this condition determines uniquely an \( x \)-dependent bare coupling constant \( g_{0}^{(N)}(x) \)

- this dependence on \( x \) can serve as an estimate of the accuracy of the calculation
**Second condition** \( b_2^{(N)}(s_2 = M^2) = 0 \)

- implies that \( b_2^{(N)}(s_2 = M^2) \) should be zero and independent of \( x \) at \( \mathbb{R}_\perp^2 \) fixed by the condition \( s_2 = M^2 \)

- these conditions should be enforced with an appropriate \( \omega \) - dependent counterterm \( Z_\omega^{(N)}(x) \) if necessary, fixed also uniquely by this condition

- the size of this counterterm can serve as an estimate of the accuracy of the calculation
How this works in practice!

$\delta m^{(n)}$ and $g_0^{(n)}$ calculated by successive solutions of the N=1, N=2 , N=3 ...N systems

- N=1 $\delta m^{(1)} = 0$
- N=2 $\delta m^{(2)}$ determined to get
  \[ \hat{P}^2 \phi(p) = M^2 \phi(p) \]
  \[ \delta m^{(2)} = -\Sigma(p^2 = M^2) \]
  $g_0^{(2)}$ determined to get
  \[ \bar{u}(k_1)\Gamma_2(s = M^2)u(p) = g_{phys}\bar{u}(k_1)u(p) \]
- etc...

This is a systematic, nonperturbative, procedure which should avoid uncancelled divergences

but also true if there are no divergences!
FIRST APPLICATIONS

Regularization with one Pauli-Villars fermion \( m_1 \) and one Pauli-Villars boson \( \mu_1 \)

QED in the two-body Fock space truncation

- should be identical to the standard result in second order of perturbation theory without any expansion in \( g \)!

- this is achieved only if the nonperturbative renormalization conditions are correctly formulated

- in that case one finds
  - \( e_0 \equiv e \) to all orders in the Fock expansion
  - anomalous magnetic moment \( \frac{\alpha}{2\pi} \) (Schwinger correction)

- at that order we have \( b_1^{(2)} = cte \) \( b_2^{(2)} = 0 \)
Yukawa model in the three-body Fock space truncation

- first non trivial nonperturbative calculation
- resummation of overlapping divergences to all orders

- calculation for $M = 0.938$ GeV and $\mu_1 = 0.138$ GeV (scalar pion)
- analytical limit to infinity taken for the Pauli-Villars fermion mass
- relative numerical accuracy of the order of $5 \times 10^{-3}$ for $\mu_1 = 10 - 1000$ GeV
- calculation done on a laptop (few hours)
anomalous magnetic moment of the physical fermion

\[ \alpha = 0.2 \]

\[ \frac{2(1 - x)Z_\omega(x)}{(2 - x)\bar{g}_0} \]

\[ \frac{\delta g_0(x)}{\bar{g}_0} \]

\rightarrow \text{correction to the calculation in the two-body Fock space truncation: 3 %}
\[ \alpha = 0.5 \]

Correction to the calculation in the two-body Fock space truncation: 9%
\( \alpha = 0.8 \)

\[ \delta g_0(x) \]

\[ \frac{2(1 - x) Z_\omega(x)}{(2 - x) \bar{g}_0} \]

\[ \frac{\delta g_0(x)}{\bar{g}_0} \]

\( \text{correction to the calculation in the two-body Fock space truncation: 15 \%} \)
regularization-scale invariance realized in our framework with an accuracy better than $5 \times 10^{-3}$ even for $\alpha = 1$.

This regularization-scale invariance does allow to make predictions for physical observables at each order of the Fock expansion and test the convergence of the Fock expansion.

One needs however to investigate the origin of the non-zero $\omega$-dependent counterterm and the $x$-dependence of the bare coupling constant!
In the light-front quenched approximation, still contributions from antiparticles are necessary in order to get full equivalence with Feynman perturbation theory (resummed to all orders). The role of antiparticle degrees of freedom is crucial for the N=3 calculation.

Irreducible first order contributions to consider in the eigenvalue equation are essential.

- Necessary in order to get full equivalence with Feynman perturbation theory (resummed to all orders).
Anomalous magnetic moment

\[ \alpha = 0.5 \]

\[ \frac{2(1 - x) Z_\omega(x)}{(2 - x) \bar{g}_0} \]

\[ \frac{\delta g_0(x)}{\bar{g}_0} \]

but slight regularization-scale non invariance for higher \( \alpha \): this first order contribution does not exhaust all contributions necessary to restore equivalence with Feynman perturbation theory.
Electromagnetic form factor $F_1(Q^2)$

$\mu_1 = 50 \text{ GeV}$

$\alpha = 0.5$

$\mu_1 = 200 \text{ GeV}$
CONCLUSIONS ....

- Well defined and controlled nonperturbative framework
  - Regularization scale invariance order by order in the Fock expansion
  - Check with results in perturbation theory (resummed to all orders) in the two-body Fock space truncation
  - Non trivial nonperturbative calculations in the three-body Fock space truncation
  - Calculation of physical observables
Systematic regularization scheme with no infinities at all, and no non-physical particles like Pauli-Villars particles

- Taylor-Lagrange regularization scheme based on the properties of fields as distributions acting on test functions with specific mathematical properties

- Application to chiral perturbation theory
  - Light front chiral effective field theory based on the Fock expansion of the state vector

- Systems with spontaneous symmetry breaking in a Fock expansion on the light-front
  - Zero modes in 3+1 dimensions in the $\lambda \Phi^4$ model

Thank you for your attention!