High-energy amplitudes in gauge theories in the next-to-leading order

G. A. Chirilli

LPT d’Orsay & CPHT

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DGLAP vs. BFKL.
Light-cone OPE versus OPE in color dipoles.
High-energy scattering and Wilson lines.
Factorization in rapidity: Feynman diagrams in a shock-wave background.
Leading order and NLO BK equation.
NLO BK kernel in $\mathcal{N} = 4$ SYM and in QCD.
NLO amplitude in $\mathcal{N}=4$.
Conclusions.
Outlook.
$x_B \sim \frac{Q^2}{s}$, \quad $\Delta x_\perp \sim \frac{1}{Q}$

Resolution of $\gamma^*$ in transverse direction (Breit frame)

**BFKL:** Balitsky, Fadin, Kuraev, Lipatov

**DGLAP:** Dokshitzer, Gribov, Lipatov, Altarelli, Parisi
BFKL: Leading Logarithmic Approximation

\[ \alpha_s \ll 1 \quad (\alpha_s \ln s)^n \sim 1 \]

\[ pQCD \text{ at LLA:} \quad A(s, t) \propto s^\Delta \]
BFKL: Leading Logarithmic Approximation \( \alpha_s << 1 \) \((\alpha_s \ln s)^n \sim 1\)

- pQCD at LLA: \( A(s, t) \propto s^\Delta \)
- Froissart-Martin theorem: \( A(s, t) \propto \ln^2 s \)
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At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross sections. **Unitarity is restored!**
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- At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross sections. **Unitarity is restored!**

- In order to take into account recombination of gluons the evolution equation for the structure function has to be non-linear.
Incoherent Interactions

Bjorken Limit

\[ Q^2 \rightarrow \infty, \ s \rightarrow \infty \]

\[ x_B = \frac{Q^2}{s} \text{ fixed} \]

resum \[ \alpha_s \ln \frac{Q^2}{\Lambda_{QCD}} \]
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Bjorken Limit

Coherent Interactions

\[ Q^2 \text{ fixed, } s \to \infty \]
\[ x_B = \frac{Q^2}{s} \to 0 \]
\[ \text{resum } \alpha_s \ln \frac{1}{x_B} \]

Regge Limit
The DIS amplitude is given by the $T$ product of two electromagnetic currents evaluated in the hadronic state

$$W^{\mu\nu} \propto \langle B | T \{ j^\mu (x) j^{\nu'} (y) | B \rangle$$

We study instead the $T \{ j^\mu (x) j^{\nu'} (y) \}$ in a generic external field.
Operator product expansion

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We study instead the \( T\{j^\mu(x)j^\nu(y)\} \) in a generic external field.

**Bjorken limit: OPE in light-ray operators**

\[
T\{j_\mu(x)j_\nu(y)\} = \frac{(x - y)^\xi}{2\pi^2(x - y)^4} \left[ 1 + \frac{\alpha_s}{\pi} \left( \ln(x - y)^2 \mu^2 + C \right) \right] \bar{\psi}(x)\gamma_\mu\gamma_\xi\gamma_\nu[x, y]\psi(y) + O\left(\frac{1}{(x - y)^2}\right)
\]
Operator product expansion

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\]

**High-energy (Regge) limit: OPE in Wilson lines operators**

\[
T\{j^\mu(x)j^\nu(y)\} = \int d^2z_1 d^2z_2 \ I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\} + \ldots
\]
Light-cone expansion and DGLAP evolution in the NLO

[Diagram of a physical process involving quark lines and gluon exchanges]
Light-cone expansion and DGLAP evolution in the NLO

\[ k_\perp^2 > \mu^2 \]

\[ k_\perp^2 < \mu^2 \]

\( \mu^2 \) - factorization scale (normalization point)

\( k_\perp^2 > \mu^2 \) - coefficient functions

\( k_\perp^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))
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OPE in light-ray operators

\[
T\{j_\mu(x)j_\nu(y)\} = \frac{(x-y)\xi}{2\pi^2(x-y)^4} \left[ 1 + \frac{\alpha_s}{\pi} \left( \ln(x-y)^2 \mu^2 + C \right) \right] \bar{\psi}(x)\gamma_\mu\gamma_\xi\gamma_\nu [x, y] \psi(y)
\]

$[x, y] \equiv Pe^{ig\int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)}$ - gauge link
Light-cone expansion and DGLAP evolution in the NLO

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Renorm-group equation for light-ray operators \( \Rightarrow \) DGLAP evolution of parton densities

\( (x - y)^2 = 0 \)

\[ \mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x, y] \psi(y) = K_{\text{LO}} \bar{\psi}(x)[x, y] \psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x, y] \psi(y) \]
High-energy expansion in color dipoles in the NLO
High-energy expansion in color dipoles in the NLO

\[ \eta > Y \quad \text{and} \quad \eta < Y \]

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High-energy evolution in gauge theories
High-energy expansion in color dipoles in the NLO

\[ Y > \eta \]

\[ Y < \eta \]

\( \eta \) - rapidity factorization scale

Rapidity \( Y > \eta \) - coefficient function ("impact factor")

Rapidity \( Y < \eta \) - matrix elements of (light-like) Wilson lines with rapidity divergence cut by \( \eta \)

\[
U_{x}^{\eta} = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} dx^{+} A_{+}^{\eta}(x_{+}, x_{\perp}) \right]
\]

\[
A^{\eta}_{\mu}(x) = \int \frac{d^{4}k}{(2\pi)^{4}} \theta(e^{\eta} - |\alpha_k|) e^{-ik \cdot x} A_{\mu}(k)
\]
The high-energy operator expansion is

\[ T\{j_\mu(x)j_\nu(y)\} = \int d^2z_1 d^2z_2 \, I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_z^\eta \hat{U}^\dagger_z^\eta\} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \, I^{\text{NLO}}_{\mu\nu}(z_1, z_2, z_3, x, y)[\text{tr}\{\hat{U}_z^\eta \hat{U}^\dagger_{z_3}^\eta\}\text{tr}\{\hat{U}_z^\eta \hat{U}^\dagger_{z_2}^\eta\} - N_c\text{tr}\{\hat{U}_z^\eta \hat{U}^\dagger_{z_1}^\eta\}] \]

In the leading order the impact factor is Möbius invariant.

In the NLO one should also expect conf. invariance since \( I^{\text{NLO}}_{\mu\nu} \) is given by tree diagrams.
High-energy expansion in color dipoles in the NLO

\[ d \frac{d}{d\eta} \text{tr}\{U_x^n U_y^{+n}\} = \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x - y)^2}{(x - z)^2 (y - z)^2} [\text{tr}\{U_x^n U_y^{+n}\} \text{tr}\{U_x^n U_y^{+n}\}] - N_c \text{tr}\{U_x^n U_y^{+n}\} + \alpha_s K_{\text{NLO}} \text{tr}\{U_x^n U_y^{+n}\} + O(\alpha_s^2) \]

\[ K_{\text{NLO}} = ? \]

(Linear part of \( K_{\text{NLO}} = K_{\text{NLO BFKL}} \))
High-energy expansion in color dipoles in the NLO

\[ A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr} \{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle + \ldots \]

\[ U(x_\perp) = P e^{ig \int_{-\infty}^{\infty} du \ n^\mu A_\mu(un+x_\perp)} \quad \text{Wilson line} \]
Consider a quark propagating at high energy in an external field ⇒ quark propagator reduces to the Wilson line collinear to quark-velocity.
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DIS at high energy: eikonal approximation (neglect recoil of gluons)

\[
A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle + \ldots
\]
\[ p_\mu p_1^\mu \equiv p_1 \cdot = \sqrt{s/2} \ p^- \]
\[ p_\mu p_2^\mu \equiv p_* = \sqrt{s/2} \ p^+ \]
Spectator frame

\[ p_\mu p_1^\mu \equiv p_1^\bullet = \sqrt{s/2} \ p^- \quad \quad p_\mu p_2^\mu \equiv p^\ast = \sqrt{s/2} \ p^+ \]

Under a Lorentz boost in the longitudinal \( z \) direction the components \( p^\mu \) get rescaled

\[ p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp \rightarrow \lambda \alpha p_1^\mu + \frac{1}{\lambda} \beta p_2^\mu + p_\perp \]
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Rescaling of the field \( A^\mu(x) \)

\[
\begin{align*}
B_\bullet(x_\bullet, x_\star, x_\perp) &= \lambda A_\bullet\left(\frac{x_\bullet}{\lambda}, x_\star \lambda, x_\perp\right) \\
B_\star(x_\bullet, x_\star, x_\perp) &= \frac{1}{\lambda} A_\star\left(\frac{x_\bullet}{\lambda}, x_\star \lambda, x_\perp\right) \\
B_\perp(x_\bullet, x_\star, x_\perp) &= A_\perp\left(\frac{x_\bullet}{\lambda}, x_\star \lambda, x_\perp\right)
\end{align*}
\]
Spectator frame

\[ p_\mu p^\mu_1 \equiv p_1 \cdot = \sqrt{s/2} p^- \quad p_\mu p^\mu_2 \equiv p_* = \sqrt{s/2} p^+ \]

Under a Lorentz boost in the longitudinal \( z \) direction the components \( p^\mu \) get rescaled

\[ p^\mu = \alpha p^\mu_1 + \beta p^\mu_2 + p_\perp \rightarrow \lambda \alpha p^\mu_1 + \frac{1}{\lambda} \beta p^\mu_2 + p_\perp \]

Rescaling of the field \( A^\mu(x) \)

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\begin{align*}
B_\bullet(x_\bullet, x_*, x_\perp) &= \lambda A_\bullet \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right) \\
B_\ast(x_\bullet, x_*, x_\perp) &= \frac{1}{\lambda} A_\ast \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right) \\
B_\perp(x_\bullet, x_*, x_\perp) &= A_\perp \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right)
\end{align*}
\]

Rescaling of the field strength \( F^{\mu\nu}(x) \)

\[
\begin{align*}
G_\bullet(x_\bullet, x_*, x_\perp) &= \lambda F_\bullet \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right) \rightarrow \delta(x_*) G_i(x_\perp) \\
G_\ast(x_\bullet, x_*, x_\perp) &= \frac{1}{\lambda} F_\ast \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right) \rightarrow 0 \\
G_\ast(x_\bullet, x_*, x_\perp) &= F_\ast \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right) \rightarrow 0 \\
G_{ik}(x_\bullet, x_*, x_\perp) &= F_{ik} \left( \frac{x_\bullet}{\lambda}, x_*, \lambda, x_\perp \right) \rightarrow 0
\end{align*}
\]
Propagation in the shock wave: Wilson line (Spectator frame)
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Each path is weighted with the gauge factor $P e^{i g \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
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$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = Pe^{ig \int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)}$$

$$p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$
Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction ⇒ we can replace the gauge factor along the actual path with the one along the straight-line path.
Each path is weighted with the gauge factor $P e^{ig \int dx_{\mu} A_{\mu}}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
\[
T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \, I_{\mu\nu}^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}
\]
\[
+ \int d^2z_1 d^2z_2 d^2z_3 \, I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3) \left[\text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_3}\} \text{tr}\{\hat{U}^\eta_{z_3} \hat{U}^{\dagger \eta}_{z_2}\} - N_c \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}\right]
\]
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\[ + \int d^2 z_1 d^2 z_2 d^2 z_3 I_{\mu\nu}^{NLO}(z_1, z_2, z_3) \left[ \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^{\dagger}\} \text{tr}\{\hat{U}_{z_2} \hat{U}_{z_3}^{\dagger}\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^{\dagger}\} \right] \]

**LO Impact Factor diagram:** \( I^{LO} \)

**NLO Impact Factor diagrams:** \( I^{NLO} \)
Conformal vectors:

\[
\kappa = \frac{\sqrt{s}}{2x_*} \left( \frac{p_1}{s} - x^2 p_2 + x_\perp \right) - \frac{\sqrt{s}}{2y_*} \left( \frac{p_1}{s} - y^2 p_2 + y_\perp \right)
\]

\[
\kappa' = \frac{\sqrt{s}}{2x'_*} \left( \frac{p_1}{s} - x'^2 p_2 + x'_\perp \right) - \frac{\sqrt{s}}{2y'_*} \left( \frac{p_1}{s} - y'^2 p_2 + y'_\perp \right)
\]

\[
\zeta_1 = \sqrt{s} \left( \frac{p_1}{s} + z_1^2 p_2 + z_1\perp \right), \quad \zeta_2 = \sqrt{s} \left( \frac{p_1}{s} + z_2^2 p_2 + z_2\perp \right)
\]

Here \( x^2 = -x_\perp^2, \quad x'^2 = -x'_\perp^2 \) (similarly for \( y \))

\[
I^{LO} \propto \frac{2}{\pi^6} \int d^2z_1\perp d^2z_2\perp \text{tr} \{ U_{z_1\perp} U_{z_2\perp}^\dagger \} \frac{z_{12\perp}^2}{x_*^2 y_*^2 (\kappa \cdot \zeta_1) (\kappa \cdot \zeta_2)^3}
\]

\[
\times \frac{\partial^2}{\partial x^{\mu} \partial y^{\nu}} \left[ -2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) + \kappa^2 (\zeta_1 \cdot \zeta_2) \right]
\]
\[ Z_3 \equiv \frac{(x-z_3)^2}{x^+} - \frac{(y-z_3)^2}{y^+} \]

\[ I^\text{NLO}_{\mu\nu}(x, y; z_1, z_2, z_3; \eta) = - I^\text{LO}_{\mu\nu} \times \frac{\alpha_s}{2\pi} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \ln \frac{\sigma s}{4} Z_3 + \text{conf.} \]

The NLO impact factor is not Möbius invariant \( \Rightarrow \) the color dipole with the cutoff \( \eta = \ln \sigma \) is not invariant.
The NLO impact factor is not Möbius invariant ⇒ the color dipole with the cutoff $\eta = \ln \sigma$ is not invariant.

However, if we define a composite operator ($a$ - analog of $\mu^{-2}$ for usual OPE)

$$ \left[ \text{Tr}\{\hat{U}\eta_{z_1} \hat{U}^\dagger\eta_{z_2} \} \right]^{\text{conf}} = \text{Tr}\{\hat{U}\eta_{z_1} \hat{U}^\dagger\eta_{z_2} \} $$

$$ + \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \frac{1}{N_c} \text{tr}\{\hat{U}\eta_{z_1} \hat{U}^\dagger\eta_{z_3} \} \text{tr}\{\hat{U}\eta_{z_3} \hat{U}^\dagger\eta_{z_2} \} - \text{Tr}\{\hat{U}\eta_{z_1} \hat{U}^\dagger\eta_{z_2} \} \right] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2) $$

the impact factor becomes conformal in the NLO.
$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \ I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{tr}[\{\hat{U}_{z_1}^\eta \hat{U}^\dagger_{z_2}^\eta\}]_{\text{conf}}$

$+ \int d^2z_1 d^2z_2 d^2z_3 \ I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}^\dagger_{z_3}^\eta\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_z^\dagger_{z_2}^\eta\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^\dagger\} \right]$
Operator expansion in conformal dipoles

\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \ I_{\mu\nu}^{LO}(z_1, z_2, x, y) \text{tr}[\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\}]^{\text{conf}} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \ I_{\mu\nu}^{NLO}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_3}\} \text{tr}\{\hat{U}^\eta_{z_3} \hat{U}^{\dagger \eta}_{z_2}\} - \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2}\} \right] \]

\[ I_{\mu\nu}^{NLO} = - I_{\mu\nu}^{LO} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a_s^2}{z_{13}^2 z_{23}^2} Z_3^2 + \text{conf.} \]

The new NLO impact factor is conformally invariant.
Operator expansion in conformal dipoles

\[
T\{j_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \, I^{LO}_{\mu\nu}(z_1, z_2, x, y) \text{tr}\{\hat{U}_{\eta}^{z_1} \hat{U}^\dagger_{\eta}^{z_2}\}^{\text{conf}}
+ \int d^2z_1 d^2z_2 d^2z_3 \, I^{NLO}_{\mu\nu}(z_1, z_2, z_3, x, y) \left[\frac{1}{N_c} \text{tr}\{\hat{U}_{\eta}^{z_1} \hat{U}^\dagger_{\eta}^{z_3}\} \text{tr}\{\hat{U}_{\eta}^{z_3} \hat{U}^\dagger_{\eta}^{z_2}\} - \text{tr}\{\hat{U}_{\eta}^{z_1} \hat{U}^\dagger_{\eta}^{z_2}\}\right]
\]

\[
I^{NLO}_{\mu\nu} = - I^{LO}_{\mu\nu} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a^2}{z_{13}^2 z_{23}^2} \, Z_3^2 + \text{conf}.
\]

The new NLO impact factor is conformally invariant.

In conformal \(\mathcal{N} = 4\) SYM theory (where the \(\beta\)-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.
Operator expansion in conformal dipoles

\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \ I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{tr}[\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \ I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) [\frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \]

\[ I_{\mu\nu}^{\text{NLO}} = - I_{\mu\nu}^{\text{LO}} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2} \frac{z_{23}^2}{z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a_s^2}{z_{13}^2 z_{23}^2} Z_3^2 + \text{conf}. \]

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N} = 4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

**Analogy:**

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counterterms order by order in perturbation theory.
Regularization of the rapidity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[ \int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty \]
Regularization of the rapidity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[ \int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty \]

Regularization by: slope

\[ U^n(x_\perp) = \text{Pexp}\left\{ ig \int_{-\infty}^{\infty} du \ n_\mu \ A^\mu (un + x_\perp) \right\} \]

\[ n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu \]
Regularization of the rapidity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[ \int_{0}^{\infty} \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty \]

Regularization by: slope

\[ U_\eta^{\mu}(x_\perp) = \text{Pexp}\left\{ ig \int_{-\infty}^{\infty} du \, n_\mu \, A_\mu^{\eta}(un + x_\perp) \right\} \quad n^{\mu} = p_1^{\mu} + e^{-2\eta} p_2^{\mu} \]

Regularization by: Rigid cut-off (used in NLO)

\[ U_x^{\eta} = \text{Pexp}\left[ ig \int_{-\infty}^{\infty} du \, p_1^{\mu} A_\mu^{\eta}(up_1 + x_\perp) \right] \]

\[ A_\mu^{\eta}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{\theta(e^{\eta} - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)}{e_\eta - |\alpha_k|} \]

\[ k^{\mu} = \alpha_k p_1^{\mu} + \beta_k p_2^{\mu} + k_\perp^{\mu} \]
\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger}\} \Rightarrow \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger}\} \rangle
\]
\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger}\} \Rightarrow \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger}\rangle \]

To get the evolution equation, consider the dipole with the rapidities up to \( \eta_1 \) and integrate over the gluons with rapidity \( \eta_1 > \eta > \eta_2 \). This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to \( \eta_2 \)).

In the frame \( || \) to \( \eta_1 \) the gluons with \( \eta < \eta_1 \) are seen as pancake.

Particles with different rapidity perceive each other as Wilson lines.
Leading order: BK equation

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \ \Rightarrow
\]

\[
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}
\]
Leading order: BK equation

\[ \frac{d}{d\eta} \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\mathrm{LO}} \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \Rightarrow \]

\[ \frac{d}{d\eta} \langle \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\mathrm{LO}} \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} \]

\[ x_\bullet = \sqrt{\frac{s}{2}} x^- \]
\[ x_* = \sqrt{\frac{s}{2}} x^+ \]
Non linear evolution equation: BK equation

\[ U_{ab}^{z} = \text{Tr}\{ t^a U_z t^b U_z^\dagger \} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s (\eta_1 - \eta_2) (U_x U_y^\dagger U_z U_y^\dagger)^{\eta_2} \]
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\} \]


\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}}\text{Tr}\{\hat{U}(x_{\perp})\hat{U}^{\dagger}(y_{\perp})\} \]


\[
\frac{d\hat{U}(x, y)}{d\eta} = \frac{\alpha_{s}N_{c}}{2\pi^{2}} \int \frac{d^{2}z}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} 
\]


LLA for DIS in pQCD \(\Rightarrow\) BFKL  
(\text{LLA: } \alpha_{s} \ll 1, \alpha_{s}\eta \sim 1)
Non linear evolution equation: BK equation

\[ U^{ab}_{z} = \text{Tr}\{t^{a}U_{z}^{\dagger}t^{b}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}}\text{Tr}\{\hat{U}(x_{\perp})\hat{U}^{\dagger}(y_{\perp})\} \]


\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_{s}N_{c}}{2\pi^{2}} \int \frac{d^{2}z (x - y)^{2}}{(x - z)^{2}(y - z)^{2}} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]

**Alternative approach:** JIMWLK (1997-2000)

LLA for DIS in pQCD \(\Rightarrow\) BFKL

LLA for DIS in sQCD \(\Rightarrow\) BK eqn

(LLA: \(\alpha_{s} \ll 1, \alpha_{s}\eta \sim 1\))

(LLA: \(\alpha_{s} \ll 1, \alpha_{s}\eta \sim 1, \alpha_{s}^{2}A^{1/3} \sim 1\))

(s for semiclassical)
Non-linear evolution equation in the NLO

\[
\frac{d}{d\eta} Tr\{U_x U_y^\dagger\} = \\
\int \frac{d^2 z}{2\pi^2} \left( \alpha_s \frac{(x - y)^2}{(x - z)^2(z - y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) [Tr\{U_x U_y^\dagger\} Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] + \\
\alpha_s^2 \int d^2 z d^2 z' \left( K_4(x, y, z, z')\{U_x, U_z^\dagger, U_y, U_y^\dagger\} + K_6(x, y, z, z')\{U_x, U_z^\dagger, U_z^\dagger, U_y, U_y^\dagger\} \right)
\]

\( K_{NLO} \) is the next-to-leading order correction to the dipole kernel and \( K_4 \) and \( K_6 \) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.
Definition of the NLO kernel

The NLO kernel is obtained in the same way as the NLO DGLAP kernel:
1. Write down the general form of the operator equation

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \left[ \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger \ldots\} \right] + O(\alpha_s^3)
\]
Definition of the NLO kernel

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\[
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\]

\[
\alpha_s^2 K_{\text{NLO}} \left[ \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger \ldots \} \right] = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)
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Definition of the NLO kernel

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\]

\[
\alpha_s^2 K_{\text{NLO}} \left[ \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\ldots\} \right] = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + O(\alpha_s^3)
\]

2. Calculate the “matrix element” of the r.h.s. in the shock-wave background

\[
\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\}\rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)
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The NLO kernel is obtained in the same way as the NLO DGLAP kernel:

1. Write down the general form of the operator equation

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\]

\[
\alpha_s^2 K_{\text{NLO}} \left[ \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\ldots\} \right] = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)
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\]

3. Subtract the LO contribution

\[
\Rightarrow \quad \left[ \frac{1}{v} \right]_+ \text{ prescription in the integrals over Feynman parameter } v
\]

Typical integral

\[
\int_0^1 dv \frac{1}{(k-p)^2_\perp v + p^2_\perp (1-v)} \left[ \frac{1}{v} \right]_+ = \frac{1}{p^2_\perp} \ln \frac{(k-p)^2_\perp}{p^2_\perp}
\]
Diagrams of the NLO gluon contribution

Add an extra gluon to the leading order diagrams

(a) (b) (c) (d)
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction

(XXXI)  (XXXII)  (XXXIII)  (XXXIV)
Diagrams of the NLO gluon contribution

"Running coupling" diagrams
Diagrams of the NLO gluon contribution

1 → 2 dipole transition diagrams

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)

G. A. Chirilli (LPT d’Orsay & CPHT)
High-energy evolution in gauge theories
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\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left( \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right)
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] - \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} \right. \\
- \left. \frac{\alpha_s N_c}{2\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\}
\]

\[
\frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_{z'} U_z^\dagger\} \right.
\]
\[
- (z' \to z) \left[ \frac{1}{(z - z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2(z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
+ \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'} U_y^\dagger U_z U_{z'} U_y^\dagger\} \right. \left( z' \to z \right) \right]
\]
\[
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \right\}
\]

Our result agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

It respects unitarity
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left( \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right) \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_z'^\dagger\} \{U_z' U_y^\dagger\} - \text{Tr}\{U_x U_z'^\dagger U_z'^\dagger U_y^\dagger U_z U_z'^\dagger\} \right\} \\
- (z' \rightarrow z) \left\{ \frac{1}{(z - z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2(z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
+ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_z'^\dagger\} \{U_z' U_y^\dagger\} - \text{Tr}\{U_x U_z'^\dagger U_z'^\dagger U_y^\dagger U_z U_z'^\dagger\} - (z' \rightarrow z) \right\} \\
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X'^2 Y^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\} \\
\]

NLO kernel = Running coupling terms + Non-conformal term + Conformal term
${\cal N} = 4$ SYM diagrams (scalar and gluino loops)
Evolution equation for color dipoles in $\mathcal{N} = 4$

( I. Balitsky and G.A.C. 2009)

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2}\} = \frac{\alpha_s}{\pi^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}z_{23}} \left\{ 1 - \frac{\alpha_sN_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
\times \left[ \text{Tr}\{T^a\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_3} T^a\hat{U}^\eta_{z_3} \hat{U}^\dagger_{z_2}\} - N_c \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2}\}\right]
\]

\[
- \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2z_3 d^2z_4}{z_{34}^4} \frac{z_{12}^2}{z_{13}z_{24}} \frac{z_{12}^2}{z_{13}^2z_{24}^2} \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \times \text{Tr}\{[T^a, T^b]\hat{U}^\eta_{z_1} T^a' T^b' \hat{U}^\dagger_{z_2} + T^b T^a \hat{U}^\eta_{z_1} [T^{b'}, T^{a'}] \hat{U}^\dagger_{z_2}\}(\hat{U}^\eta_{z_3})^{aa'} (\hat{U}^\eta_{z_4} - \hat{U}^\eta_{z_3})^{bb'}
\]

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.
Evolution equation for color dipoles in $N = 4$

(I. Balitsky and G.A.C. 2009)

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_1^\eta \hat{U}_2^\eta\}$$

$$= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{22}^2} \right] \right\}$$

$$\times \left[ \text{Tr}\{T^a \hat{U}_1^\eta \hat{U}_3^\eta T^a \hat{U}_3^\eta \hat{U}_2^\eta\} - N_c \text{Tr}\{\hat{U}_1^\eta \hat{U}_2^\eta\} \right]$$

$$- \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}$$

$$\times \text{Tr}\{[T^a, T^b] \hat{U}_1^\eta T^b T^a \hat{U}_2^\eta + T^b T^a \hat{U}_3^\eta [T^b', T^a'] \hat{U}_2^\eta\} (\hat{U}_3^\eta)^{aa'} (\hat{U}_4^\eta - \hat{U}_3^\eta)^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant.
Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$ SYM

\[
[\text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}]^{\text{conf}} = \text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}
\]

\[+ \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger \eta}\} - \text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}\right] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)
\]

\[
\frac{d}{d\eta} [\text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}]^{\text{conf}}
\]

\[= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 - \frac{\alpha_s N_c \pi^2}{4\pi} \frac{1}{3} \right] \left[ \text{Tr}\{T^a \hat{U}_z^\eta \hat{U}_z^{\dagger \eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger \eta}\} - N_c \text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}\right]^{\text{conf}}
\]

\[\quad - \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{23}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 z_{23}^2} + \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\}
\]

\[\times \text{Tr}\{[T^a, T^b] \hat{U}_z^\eta T^{a'} T^{b'} \hat{U}_z^{\dagger \eta} + T^b T^a \hat{U}_z^\eta [T^{b'}, T^{a'}] \hat{U}_z^{\dagger \eta}\}((\hat{U}_z^\eta)^{aa'} (\hat{U}_z^{\eta})^{bb'} - (z_4 \rightarrow z_3))
\]

Now Möbius invariant!
To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{U}^n(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x^n \hat{U}_y^n\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{U}_{\text{conf}}^n(z_1, z_2) = \hat{U}^n(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} [\hat{U}^n(z_1, z_3) + \hat{U}^n(z_2, z_3) - \hat{U}^n(z_1, z_2)]$$
NLO BFKL equation in $\mathcal{N} = 4$ SYM

To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{U}^\eta(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger \eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{U}^\eta_{\text{conf}}(z_1, z_2) = \hat{U}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\hat{U}^\eta(z_1, z_3) + \hat{U}^\eta(z_2, z_3) - \hat{U}^\eta(z_1, z_2)]$$

NLO BFKL

$$\frac{d}{d\eta} \hat{U}^\eta_{\text{conf}}(z_1, z_2)$$

$$= \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 - \frac{\alpha_s N_c}{4\pi^2} \frac{\pi^2}{3} \right] [\hat{U}^\eta_{\text{conf}}(z_1, z_3) + \hat{U}^\eta_{\text{conf}}(z_2, z_3) - \hat{U}^\eta_{\text{conf}}(z_1, z_2)]$$

$$+ \frac{\alpha_s^2 N_c^2}{8\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{23}^2 z_{14}^2 z_{24}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{23}^2} + \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \hat{U}^\eta_{\text{conf}}(z_3, z_4)$$

$$+ \frac{3\alpha_s^2 N_c^2}{2\pi^3} \zeta(3) \hat{U}^\eta_{\text{conf}}(z_1, z_2)$$

Eigenvalues agree with Kotikov and Lipatov (2000)
NLO evolution of composite “conformal” dipoles in QCD

\[
\frac{d}{d\eta} [\text{tr}\{\hat{U}_{z_1} U_{z_2}^\dagger\}]^{\text{conf}} = \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left( [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} \times \frac{z_1^2}{z_2^2 z_3^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( b \ln \frac{z_1^2}{\mu^2} + b \frac{z_1^2 - z_2^2}{z_2^2 z_3^2} \ln \frac{z_1^2}{z_2^2} + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right)
\]

\[
+ \frac{\alpha_s}{4\pi^2} \int d^2 z_4 \frac{z_4^2}{z_3^4} \left\{ -2 + \frac{z_1^2 z_3^2 z_4^2}{2(z_1^2 z_3^2 - z_2^2 z_4^2)} \ln \frac{z_1^2 z_3^2}{z_2^2 z_4^2} \right\}
\]

\[
+ \frac{z_1^2 z_3^2 z_4^2}{z_2^2 z_1^2 z_4^2} \left[ 2 \ln \frac{z_1^2 z_3^2}{z_2^2 z_4^2} + \left( 1 + \frac{z_2^2 z_4^2}{z_1^2 z_3^2} \right) \ln \frac{z_1^2 z_3^2}{z_4^2 z_2^2} \right]
\]

\[
\times \left\{ \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_4}^\dagger \hat{U}_{z_3} \hat{U}_{z_2}^\dagger \hat{U}_{z_4} \hat{U}_{z_3}^\dagger\} - (z_4 \rightarrow z_3) \right\}
\]

\[
b = \frac{11}{3} N_c - \frac{2}{3} n_f
\]

\[
K_{\text{NLO BK}} = \text{Running coupling part} + \text{Conformal "non-analytic" (in j) part} + \text{Conformal analytic (N = 4) part}
\]

Linearized \(K_{\text{NLO BK}}\) reproduces the known result for the forward NLO BFKL kernel.
Small-$x$ (Regge) limit in the coordinate space

\[(x - y)^4 (x' - y')^4 \langle \mathcal{O}(x) \mathcal{O}^\dagger(y) \mathcal{O}(x') \mathcal{O}^\dagger(y') \rangle\]

Regge limit: \(x_+ \rightarrow \rho x_+, \ x'_+ \rightarrow \rho x'_+, \ y_- \rightarrow \rho' y_-, \ y'_- \rightarrow \rho' y'_- \quad \rho, \rho' \rightarrow \infty\)

Regge limit symmetry in a conformal theory: 2-dim conformal Möbius group \(SL(2, \mathbb{C})\).
Small-$x$ (Regge) limit in the coordinate space

\[(x - y)^4 (x' - y')^4 \langle \mathcal{O}(x) \mathcal{O}^\dagger(y) \mathcal{O}(x') \mathcal{O}^\dagger(y') \rangle\]

Regge limit: \(x_+ \to \rho x_+, x'_+ \to \rho x'_+, y_- \to \rho' y_-, y'_- \to \rho' y_-\) \(\rho, \rho' \to \infty\)

**LLA:** \(\alpha_s \ll 1, \alpha_s \ln \rho \sim 1, \Rightarrow \sum (\alpha_s \ln \rho)^n \equiv \text{BFKL pomeron.} \)
LLA \(\Leftrightarrow\) tree diagrams \(\Rightarrow\) the BFKL pomeron is Möbius invariant.

**NLO LLA:** extra \(\alpha_s:\sum \alpha_s (\alpha_s \ln \rho)^n \equiv \text{NLO BFKL}\)
In a conformal theory \((\mathcal{N} = 4 \text{ SYM})\) we expect NLO BFKL to be Möbius invariant.
Small-\(x\) (Regge) limit in the coordinate space

\[(x - y)^4(x' - y')^4 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y') \rangle\]

Regge limit: \(x_+ \to \rho x_+, x'_+ \to \rho x'_+, y_- \to \rho' y_-, y'_- \to \rho' y_-\) \(\rho, \rho' \to \infty\)

**LLA:** \(\alpha_s \ll 1, \alpha_s \ln \rho \sim 1, \Rightarrow \sum (\alpha_s \ln \rho)^n \equiv \text{BFKL pomeron.} \)

LLA ⇔ tree diagrams ⇒ the BFKL pomeron is Möbius invariant.

**NLO LLA:** extra \(\alpha_s\): \(\sum \alpha_s (\alpha_s \ln \rho)^n \equiv \text{NLO BFKL}\)

In a conformal theory (\(\mathcal{N} = 4\) SYM) we expect NLO BFKL to be Möbius invariant.

In QCD, we have running coupling part plus conformal part.
In a conformal theory the amplitude \( A(x, y; x', y') \) depends on two conformal ratios which can be chosen as

\[
R = \frac{(x - x')(y - y')^2}{(x - y)^2(x' - y')^2},
\]

\[
r = R \left[ 1 - \frac{(x - y')^2(y - x')^2}{(x - x')^2(y - y')^2 + \frac{1}{R}} \right]^2.
\]

In the Regge limit \( R \) scales as \( \rho^2 \rho'^2 \) while \( r \) does not depend on \( \rho \) or \( \rho' \).
In a conformal theory the amplitude $A(x, y; x', y')$ depends on two conformal ratios which can be chosen as

$$R = \frac{(x - x')(y - y')^2}{(x - y)^2(x' - y')^2}.$$ 

$$r = R \left[ 1 - \frac{(x - y')^2(y - x')^2}{(x - x')^2(y - y')^2} + \frac{1}{R} \right]^2$$

In the Regge limit $R$ scales as $\rho^2\rho'^2$ while $r$ does not depend on $\rho$ or $\rho'$. 

G. A. Chirilli (LPT d'Orsay & CPHT)
High-energy evolution in gauge theories
LPT March 04, 2010
The pomeron contribution in a conformal theory can be represented as an integral over one real variable $\nu$ 

\[
(x - y)^4 (x' - y')^4 \langle \mathcal{O}(x) \mathcal{O}^\dagger(y) \mathcal{O}(x') \mathcal{O}^\dagger(y') \rangle = \frac{i}{2} \int d\nu \tilde{f}_+(\nu) \frac{\tanh \pi \nu}{\nu} F(\nu) \Omega(r, \nu) R^{\frac{1}{2}} \omega(\nu)
\]

$\omega(\nu) \equiv \omega(0, \nu)$ is the pomeron intercept,

$\tilde{f}_+(\omega) = (e^{i\pi \omega} - 1)/\sin \pi \omega$ is the signature factor in the coordinate space.

$F(\nu)$ is the “pomeron residue”.

The conformal function $\Omega(r, \nu)$ is given by a representation in terms of the two-dimensional integral

\[
\Omega(r, \nu) = \frac{\nu^2}{\pi^3} \int d^2z \left( \frac{\kappa^2}{(2\kappa \cdot \zeta)^2} \right)^{\frac{1}{2}+i\nu} \left( \frac{\kappa'^2}{(2\kappa' \cdot \zeta)^2} \right)^{\frac{1}{2}-i\nu}
\]

where $\zeta \equiv p_1 + \frac{z_\perp}{s} p_2 + z_\perp$
Operator expansion in conformal dipoles in $\mathcal{N} = 4$ SYM

$$\mathcal{O} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2 - 1}} \text{Tr}\{Z^2\} \quad (Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)) \text{ renorm-invariant chiral primary operator}$$

$$(x - y)^4 T\{\hat{O}(x)\hat{O}^\dagger(y)\} = \frac{(x - y)^4}{\pi^2(N_c^2 - 1)} \int d^2z_1 d^2z_2 \frac{\left(x_\ast y_\ast\right)^{-2}}{Z_1^2 Z_2^2} \text{Tr}\{\hat{U}_1^\eta \hat{U}^\dagger_2^\eta\}\text{conf}$$

$$- \frac{\alpha_s (x - y)^4}{2\pi^4(N_c^2 - 1)} \int d^2z_1 d^2z_2 d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \frac{\left(x_\ast y_\ast\right)^{-2}}{Z_1^2 Z_2^2}$$

$$\times \left( \ln \frac{x_\ast y_\ast z_{12}^2 e^{2\eta}}{16(x - y)^2 z_{13}^2 z_{23}^2} \left[ \frac{(x - z_3)^2}{x_\ast} - \frac{(y - z_3)^2}{y_\ast} \right]^2 - i\pi + 2C \right)$$

$$\times \left[ \text{Tr}\{T^n \hat{U}_1^\eta \hat{U}^\dagger_3^\eta T^n \hat{U}_3^\eta \hat{U}^\dagger_2^\eta\} - N_c \text{Tr}\{\hat{U}_1^\eta \hat{U}^\dagger_2^\eta\} \right]$$

The impact factor is Möbius invariant and does not scale with the energy.
\[(x - y)^4(x' - y')^4 \langle T\{\hat{O}(x)\hat{O}(y)^\dagger \hat{O}(x')\hat{O}^\dagger(y')\}\rangle \]
\[= -\frac{1}{\pi^4} \int d\nu \int d^2z_0 \frac{1 + 4\nu^2}{8\pi} \frac{\Gamma^2(\frac{1}{2} - i\nu)}{\Gamma(1 - 2i\nu)} \left(\frac{\kappa^2}{4(\kappa \cdot \zeta_0)^2}\right)^{\frac{1}{2} + i\nu} \]
\[\times \frac{(-a_0 b_0 + i\epsilon)^{\frac{1}{2}\omega(\nu)} - (a_0 b_0 + i\epsilon)^{\frac{1}{2}\omega(\nu)}}{\pi \omega} I_0(\nu) \left[1 - \frac{\alpha_s N_c}{4\pi} \Phi_1(\nu)\right] \]
\[\times \frac{1 + 4\nu^2}{8\pi} \frac{\Gamma^2(\frac{1}{2} + i\nu)}{\Gamma(1 + 2i\nu)} \left(\frac{\kappa'^2}{4(\kappa' \cdot \zeta'_0)^2}\right)^{\frac{1}{2} - i\nu} I_0(\nu) \left[1 - \frac{\alpha_s N_c}{4\pi} \Phi_1(\nu)\right] \]
\[\times \left[1 - \frac{\alpha_s N_c}{2\pi} \left(\chi(\gamma) \left\{4C + \frac{2}{\gamma(1 - \gamma)}\right\} + \frac{\pi^2}{3}\right)\right] \]

\[F(\nu) = \frac{N_c^2}{N_c^2 - 1} \frac{16\pi^4 \alpha_s^2}{\cosh^2 \pi \nu} \left[1 - \frac{\alpha_s N_c}{4\pi} \Phi_1(\nu)\right] \left[1 - \frac{\alpha_s N_c}{4\pi} \Phi_1(\nu)\right] \]
\[\left[1 - \frac{\alpha_s N_c}{2\pi} \left(\chi(\gamma) \left\{4C + \frac{2}{\gamma(1 - \gamma)}\right\} + \frac{\pi^2}{3}\right)\right] + O(\alpha_s^2) \]

which gives the pomeron residue in the next-to-leading order.
Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in QCD and $\mathcal{N} = 4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.
- The NLO Amplitude in $\mathcal{N} = 4$ is given an integral over a real parameter $\nu$. 
Outlook

- NLO Photon Impact Factor in QCD.
- NLO amplitude of $\gamma^* \gamma^*$ scattering (QCD).
- 3-pomeron vertex in the Wilson line formalism.
- Odderon in the Wilson line formalism.
- NLO kernel for the B-JIMWLK equation.