TWO-DIMENSIONAL COULOMB SYSTEMS: SOLVABLE MODELS AT $\Gamma = 2$

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1. INTRODUCTION

At the previous conference, in Santa Cruz, A. Alastuey gave a review talk\(^1\) with a similar title. This review was centered on the equilibrium statistical mechanics of the classical two-dimensional one-component plasma; at the special value $\Gamma = 2$ of the coupling constant, many exact explicit results had been obtained for both bulk and surface properties.

Two especially tantalizing problems were unsolved at that time. First, although the one-component plasma had been solved for a large class of inhomogeneous cases, one had not succeeded in dealing with a background having the double periodicity of a two-dimensional crystal; this would have been a model for (classical) electrons in a crystalline field or for a superionic conductor. Second, although a lattice version of the two-component plasma\(^2\) had just been solved, the bulk and surface properties of the two-component plasma in the continuum had not been studied.

I am glad to report that these two problems (which actually are related) have now been solved, and more generally, that we have a better overall understanding of the two-dimensional Coulomb systems at $\Gamma = 2$, largely thanks to the fresh insight of a graduate student, François Cornu\(^3\). The present review will include the recent progress in today's perspective.

The two-dimensional Coulomb systems which we consider are made of particles of charge $\pm e$. The interaction potential between two identical particles is $-e^2 n_0(r/L)$, where $r$ is the distance between the particles and $L$ a length scale which only fixes the zero of energy. Using this logarithmic interaction, which is the two-dimensional version of the Coulomb potential, insures that our two-dimensional systems will be qualitatively very similar to the three-dimensional Coulomb systems with an $e^2/r$ interaction. In the two-dimensional case, the relevant dimensionless coupling constant is $\Gamma = e^2/k_B T$ (where $T$ is the temperature and $k_B$ is Boltzmann's constant); for point-particles, the equation of state\(^4\) has the trivial form

$$\rho = \rho_0 k_B T (1 - \frac{\Gamma}{4})$$

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where \( p \) is the pressure and \( \rho \) the number density. For the other thermodynamical quantities and the correlations, both the one-component plasma (particles of one sign in a continuous background of opposite charge) and the two-component plasma (positive and negative particles) become solvable models for the special temperature such that \( \Gamma = 2 \). We shall study these models at \( \Gamma = 2 \), starting with the two-component plasma.

2. TWO-COMPONENT PLASMA

For a long time, it had been known to field theorists that the two-dimensional two-component plasma (Coulomb gas) should be simple at \( \Gamma = 2 \), because it is then equivalent to a free Fermi field. Indeed, the Coulomb gas is equivalent \(^5\) to a Sine-Gordon field, which in turn is equivalent \(^6,7\) to the massive Thirring model; the latter one reduces to a free Fermi field when \( \Gamma = 2 \). In the following, the equivalence to a free Fermi field will be shown directly. The two-component plasma is described in the grand-canonical ensemble.

For \( \Gamma \geq 2 \), a two-component plasma made of point particles is unstable against collapse. For overcoming this difficulty, Gaudin\(^2\) introduced a lattice model and solved it at \( \Gamma = 2 \). This model is a convenient starting point.

2.1. Lattice model

We represent the position \( \mathbf{r} = (x, y) \) of a particle by the complex number \( z = x + iy \). Two interwoven lattices \( U \) and \( V \) are introduced. The positive (negative) particles sit on the sublattice \( U(V) \); each lattice site is occupied by zero or one particle. Let \( u_i(u_i \in U) \) be the complex coordinate of the \( i \)th positive particle and \( v_j(v_j \in V) \) be the complex coordinate of the \( j \)th negative particle. For one positive and one negative particle, the canonical partition function is

\[
\sum_{u_i \in U} \sum_{v_j \in V} \exp(-2\pi \ln |u_i - v_j| - \frac{\pi^2}{L}) \frac{1}{|u_i - v_j|^2} = \sum_{u_i \in U} \sum_{v_j \in V} \frac{\det \begin{bmatrix} L & u_i - v_j \\ u_i - v_j & 1 \end{bmatrix}}{u_i - v_j - 1}.
\]

A similar expression holds for \( n \) positive and \( n \) negative particles. It is convenient to introduce a more compact notation: each lattice site is characterized by its complex position \( z \) and by an isospinor, \( (\frac{1}{2}, 0) \) for a positive site, \( (0, \frac{1}{2}) \) for a negative one. In terms of Pauli matrices \( \sigma \) operating in the isospinor space, the grand partition function (with only neutral configurations) is

\[
Z = 1 + \lambda^2 \sum_{u_i \in U} \sum_{v_j \in V} \frac{1}{|u_i - v_j|^2} + \ldots,
\]
where $\lambda$ is the fugacity per site; it can be recognized as the expansion of the determinant

$$Z = \det \left[ 1 + \frac{\sigma_x i \sigma_y}{2} \frac{\lambda l}{z-z'} + \frac{\sigma_x i \sigma_y}{2} \frac{\lambda L}{z-z'} \right];$$

the lines and the columns of this determinant are labeled by the positions and charge signs of all the lattice sites.

Gaudin went on with this lattice model, and he solved it in terms of elliptic functions. A somewhat simpler, although less symmetrical, version can be obtained by constraining the particles to sit on an array of parallel equidistant lines rather than on a lattice; this parallel line model\(^8\) can be solved in terms of trigonometric functions. Still simpler however is the continuum limit.

2.2. Small charged hard discs\(^8\)

In the continuum limit, the factor $1/(z-z')$ which occurs in the grand partition function $Z$ can be viewed as the $(z,z')$ element of an infinite matrix. The key point for a simple proof of the equivalence between our Coulomb gas and a free Fermi field is that the inverse of the matrix with elements $1/(z-z')$ is the differential operator $(2\pi)^{-1} (\partial_x i \partial_y)$. Indeed

$$\frac{1}{2\pi} (\partial_x i \partial_y) \frac{1}{z-z'} = \frac{1}{2\pi} \left( \partial_x i \partial_y \right) \left( \partial_x i \partial_y \right) \ln|\mathbf{r}' - \mathbf{r}| = \delta(\mathbf{r}' - \mathbf{r}).$$

Thus, in terms of a rescaled fugacity $m = 2\pi \lambda l/S$, where $S$ is the area per site, in the continuum limit the grand partition function can be rewritten as

$$Z = \det \left[ (\sigma_x \partial_x + \sigma_y \partial_y + m)(\sigma_x \partial_x + \sigma_y \partial_y)^{-1} \right]$$

and

$$\ln Z = \text{Tr} \left[ \ln(\sigma_x \partial_x + \sigma_y \partial_y + m) - \ln(\sigma_x \partial_x + \sigma_y \partial_y) \right].$$

The fugacity $m$ becomes the mass in the two-dimensional Dirac operator $\sigma_x \partial_x + \sigma_y \partial_y + m$.

Before proceeding to the calculation of the pressure $p$ from $\ln Z$, it is convenient to discuss the one-body and n-body densities. Replacing $m$ for a while by a charge and position dependent fugacity $[(1+q_x)/2]m_+(\mathbf{r}) + [(1-q_x)/2]m_-(\mathbf{r})$ and taking functional derivatives of $\ln Z$ with respect to $m_\pm(\mathbf{r})$, one obtains the densities in terms of the propagator

$$G_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = \left< \mathbf{r}_1 \mathbf{s}_1 | \frac{1}{\sigma_x \partial_x + \sigma_y \partial_y + m} | \mathbf{r}_2 \mathbf{s}_2 \right>,$$

where $s_i = \pm 1$ is the sign of the particle at $\mathbf{r}_i$. More explicitly,

$$G_{++}(\mathbf{r}_1, \mathbf{r}_2) = G_{--}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \nu_0(m r_{12}),$$

$$G_{+-}(\mathbf{r}_1, \mathbf{r}_2) = -G_{-+}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \nu_1(m r_{12}),$$

$$G_{\pm 0}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \nu_0(m r_{12}).$$
\[ G_+(\vec{r}_1, \vec{r}_2) = G_-(\vec{r}_2, \vec{r}_1) = -\frac{m}{2\pi} \cdot e^{i\theta} K_0(m r_{12}) , \]

where \( \theta = \vec{r}_2 \cdot \vec{r}_1 \), \( \theta \) is the polar angle of \( \vec{r}_{12} \), and \( K_0 \) and \( K_1 \) are modified Bessel functions. The 2-body truncated densities are
\[
\rho_{++}^{(2)}(\vec{r}_1, \vec{r}_2) = \rho_{--}^{(2)}(\vec{r}_1, \vec{r}_2) = \rho_+^{(2)}(\vec{r}_1, \vec{r}_2) - [mG_+(\vec{r}_1, \vec{r}_2)]^2 = \frac{e^{i\theta}}{2\pi} [K_0(m r_{12})]^2 ,
\]
\[
\rho_{+-}^{(2)}(\vec{r}_1, \vec{r}_2) = \rho_{-+}^{(2)}(\vec{r}_1, \vec{r}_2) = \rho_+^{(2)}(\vec{r}_1, \vec{r}_2) - |mG_+(\vec{r}_1, \vec{r}_2)|^2 = \frac{e^{i\theta}}{2\pi} [K_1(m r_{12})]^2 ,
\]

and higher-order truncated densities can be expressed as sum of products of factors \( G_+ \) and \( G_- \). The Bessel functions behave like \( \exp(-m r_{12}) \) at large distance, and therefore the correlations have an exponential decay, with an inverse correlation length of the order of \( m \), the rescaled fugacity.

A remarkable feature of the truncated \( n \)-body densities \( (n \geq 2) \) is that they are finite, for a given value of \( m \) and for non-zero separations, even for a point-particle system. The collapse at \( \Gamma = 2 \) is however apparent on the one-body densities: for a point-particle system, the densities of positive and negative particles would be \( \rho_+ = \rho_- = mG_+(\vec{r}, \vec{r}) = (m^2/2\pi) K_0(0) = \infty \). This divergence originates in the small distance divergence of the one-pair canonical integral \( \int d^2r \left< L/r \right>^2 \). In order to suppress it, one may replace the point particles by charged hard discs of small diameter \( R \). The one-body densities can then be obtained by using the perfect-screening sum rule
\[
\frac{1}{2} \rho_0 = \rho_{+} = \int_{r > R \rho} \left( \rho_{++}^{(2)}(r) - \rho_+^{(2)}(r) \right) d^2 r ,
\]

with the result
\[
\rho_+ = \frac{m^2}{2\pi} K_0(m R) - \frac{m^2}{2\pi} \left[ \ln \frac{2}{\pi R} \gamma - \gamma - \frac{1}{2} \right] ,
\]

where \( \gamma \approx 0.5772 \ldots \) is Euler's constant. By integration with respect to \( m \), one obtains the pressure \( p \):
\[
p = \frac{1}{k_B T} \cdot \frac{m^2}{2\pi} \left[ \ln \frac{2}{\pi R} \gamma + \frac{1}{2} \right] .
\]

This behavior of the pressure can also be obtained directly by evaluating \( \ln Z \) in the momentum representation, with some high-momentum cutoff of the order of \( R^{-1} \); the precise value to be taken for the cutoff is determined by the perfect-screening rule.

The above expressions of the density and the pressure are the leading-order terms as \( R \to 0 \); the theory is valid for hard cores small in the sense \( m R < 1 \) or \( \rho R^2 < 1 \). For point particles, the equation of state becomes \( p = (1/2) \rho k_B T \), describing a gas of collapsed neutral pairs. The knowledge of the correlation functions allows the calculation of other thermodynamical quantities, such as
the excess internal energy per particle
\[ u = \frac{1}{4} e^2 \left[ \ln \frac{2R}{mL} - \gamma \right] \]
or the excess specific heat at constant volume, per particle,
\[ c_V = \frac{1}{6} \left( \ln \frac{2\pi R}{mL} - \gamma \right)^2 - \frac{1}{4} \left( \ln \frac{2\pi R}{mL} - \gamma \right) - \frac{1}{8} \]

The thermodynamical properties are close to what would be obtained for a system of independent neutral pairs. However, as \( R \neq 0 \), there are a few "free" particles which give screening effects, i.e., correlations characteristic of a conducting system: the correlations decay exponentially, and the Stillinger-Lovett second-moment sum rule
\[ 2 \int d^2 r \left[ \rho_+^2(r) - \rho_-^2(r) \right] = -\frac{2k_B T}{m e^2} = -\frac{1}{\pi} \]
is indeed satisfied. This is in agreement with the known fact that the Kosterlitz-Thouless transition to a dielectric phase occurs only at a lower temperature such that \( \Gamma \approx 4 \).

2.3. Electrical double layers

The electrical double layer which forms at the interface between two conducting media can be mimicked by inhomogeneous two-dimensional plasmas. Here we consider two-component plasmas. External potentials such as walls can be taken into account by using a charge and position dependent fugacity. The propagator has now the more general form
\[ G^{s_1 s_2}_{s_1 s_2}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sigma_{s_1}^2 + \sigma_{s_2}^2 + m_+^2(\vec{r}_1) - \frac{1}{2} \sigma_{s_1}^2 - \frac{1}{2} \sigma_{s_2}^2} \]

The one-body and n-body densities can still be expressed in terms of \( G \). In particular
\[ \rho_+(\vec{r}) = m_+(\vec{r}) G^+_{++}(\vec{r}, \vec{r}) \]
\[ \rho_-(\vec{r}) = m_-(\vec{r}) G^-_{--}(\vec{r}, \vec{r}) \]
here too we deal with small hard discs, and a cutoff has to be used for the densities to be finite.

In general, the propagator \( G \) is the solution of a system of coupled partial differential equations, which are, in a 2x2 matrix notation
\[ \begin{align*}
[\sigma_{s_1}^2 + \sigma_{s_2}^2 + m_+^2(\vec{r}_1) - \frac{1}{2} \sigma_{s_1}^2 - \frac{1}{2} \sigma_{s_2}^2] \delta(\vec{r}_1 - \vec{r}_2) = \delta(\vec{r}_1 - \vec{r}_2) \end{align*} \]
Here, we only deal with plane interfaces, and the fugacities depend only on \( x \), the coordinate in the direction normal to the interface: \( m_+^2(\vec{r}) = m_+(x) \). The standard technique then is to introduce the Fourier transform \( \tilde{G} \) with respect to \( y_{12} \), the coordinate difference along the interface:
\[ \hat{G}(x_1, x_2, z) = \sum_{i=1}^{\infty} G(x_1, x_2, y_{12}) e^{i\beta y_{12}} \]

This transformation leads to systems of two coupled ordinary differential equations, such as:

\[
\begin{align*}
& m_+ (x_1) \hat{G}_{++} (x_1, x_2, \xi) + \left( \frac{d}{dx_1} + \xi \right) \hat{G}_{+-} (x_1, x_2, \xi) = \delta(x_1 - x_2), \\
& (\frac{d}{dx_1} - \xi) \hat{G}_{+-} (x_1, x_2, \xi) + m_- (x_1) \hat{G}_{++} (x_1, x_2, \xi) = 0.
\end{align*}
\]

In the cases which have been considered, \( m_+ (x) \) and \( m_- (x) \) are constant or exponential functions with discontinuities; the differential equations for \( \hat{G} \) are easily solved by splicing exponentials together, and an inverse Fourier transformation provides a simple integral representation for \( G \) and the densities.

The following interfaces have been studied:

A. Charged hard wall (primitive electrode)

The hard wall occupies the half-space \( x < 0 \) and carries a "surface" charge density (actually charge per unit length) \( -\sigma \). In the half-space \( x > 0 \), the wall creates a constant external electrical field represented by fugacities \( m_+ (x) = m \exp (4\pi \sigma x) \). The density profiles are drawn on Fig.1. Note the drop in density near the wall: the pressure is close to half the ideal gas value, and therefore the density at the wall (which generates the pressure) has to be small, at least for a moderate surface charge.

\[ m^{-2} \rho_s (x) \]

\[ 0.00 \quad 0.25 \quad 0.50 \quad 0.75 \quad 1.00 \]

\[ mx \]

\[ 0.25 \quad 0.50 \quad 0.75 \quad 1.00 \]

FIGURE 1

The density profiles near a hard wall. For an uncharged wall (\( \sigma = 0 \)), \( \rho_+ (x) = \rho_- (x) \) (black circles). For a charged wall (\( 2\sigma = m \)), \( \rho_+ (x) \) (crosses) and \( \rho_- (x) \) (white circles). The cutoff is \( mR = 0.01 \).
B. Polarizable interface

A polarizable interface (impermeable membrane separating two media) is described by giving different values \( m^a, m^b, m'^a, m'^b \) to the fugacities in regions \( a(x>0) \) and \( b(x<0) \). Actually, the physics depends only on three parameters (the bulk density on each side and the potential drop across the interface), and only three combinations of the fugacities are relevant quantities: \( m = (m^a m'^a)^{1/2} \), \( m^b = (m^b m'^b)^{1/2} \), and \( m^a m^b / m'^a m'^b \). The density profiles are drawn on Fig.2. Note the discontinuities in the densities at the interface.

![Graph showing density profiles near a polarizable interface.](image)

**FIGURE 2**
The density profiles near a polarizable interface, for \( m^b = 0.25, m'^b = 0.25 \) and \( m^a = m'^a = 16 \): \( \rho^a(x) \) (crosses) and \( \rho^b(x) \) (white circles). The cutoff is \( mR = 0.01 \).

C. Semipermeable membrane

It is also possible to describe a membrane permeable to, say, the positive particles and impermeable to the negative ones. Now there are two relevant parameters which can be chosen independently, the bulk density on each side, or alternatively \( m = (m^a m'^a)^{1/2} \) and \( m^b = (m^b m'^b)^{1/2} \). The density profiles are drawn on Fig. 3. Note the continuity of the density of positive particles which can freely cross the membrane.
The density profiles near a semipermeable membrane, for $m_{-}^{b} m_{-}^{b} = 0.25 m_{x}^{b} m_{x}^{a}$; $\rho_{+}(x)$ (crosses) and $\rho_{-}(x)$ (white circles). The cutoff is $mR = 0.01$.

For both the charged hard wall and the polarisable interface, it is possible to compute exactly quantities of interest to the electrochemists, such as the differential capacity and the surface tension. The "surface" charge density $\varepsilon$ on the right-hand side of the interface, the potential drop $\Delta \phi$ (electrical potential at $x = +$ minus electrical potential at $x = -\infty$) and the surface tension $\gamma$ are related by the exact Lippmann equation

$$\frac{\Delta \gamma}{\Delta \phi} = -\varepsilon \sigma .$$

2.4. Two-component plasma with a background

The general propagator $G$ of section 2.3. can be used for studying a system made of positive and negative particles immersed in a continuous charged background. This could be a model for a metal-salt liquid mixture, of the type alkali metal plus halide of this metal. This model is obtained by including in the fugacities $m_{+}(\Gamma)$ and $m_{-}(\Gamma)$ the electrostatic potential $e\phi(\Gamma)$ created by the background. Explicit expressions can be obtained for bulk and surface properties of this system.

For future reference, let us note that, if there are only electrostatic interactions, the fugacities are of the form $m_{+}(\Gamma) = m \exp \left( + \frac{2\phi(\Gamma)}{kT} \right)$ and the $G_{+}$ part of the propagator can be shown to be alternatively given by
-V(\vec{r}_1) \mathcal{G}_{\alpha\alpha}(\vec{r}_1,\vec{r}_2) e^{-V(\vec{r}_2)} = \langle \vec{r}_1 | \frac{\hbar^2}{2m^* A^*} \frac{\partial}{\partial \vec{r}_2} | \vec{r}_2 \rangle,

where \( A = \bar{a}_x + i \bar{a}_y + i \bar{a}_x V(\vec{r}) + i \bar{a}_y V(\vec{r}) \).

3. ONE-COMPONENT PLASMA REVISITED

The (in general inhomogeneous) one-component plasma can be recovered \(^{11}\) by starting with a two-component plasma with a, say, negative background, and letting the fugacity \( m \) in the previous formula go to zero. Then, the negative particles disappear, and only those positive particles which are necessary for neutralizing the background remain: one obtains a canonical description of the one-component plasma.

In this limit \( m \to 0 \), the operator \( m^2 / (m^2 + A^* A) \) becomes the projector \( P \) on the eigenstates of \( A^* A \) with eigenvalue zero, i.e., the projector on the space \( E \) of the functions of the form \( \psi = \exp[-V(\vec{r})] \) times an entire function of \( z = x + iy \).

The densities are

\( \rho(\vec{r}) = \langle \vec{r} | P | \vec{r} \rangle, \rho^{(2)}(\vec{r}_1,\vec{r}_2) = -\langle \vec{r}_1 | P | \vec{r}_2 \rangle^2, \) etc...

and the one-component plasma problem amounts to compute the projector \( P \), i.e., to find an orthogonal basis for the space \( E \).

Alternatively, the same can be shown by starting directly with the canonical Boltzmann factor of the one-component plasma \(^{12}\). The Hamiltonian of \( N \) charges \( e \) is

\[ H = e^2 \sum \limits_{i=1} V(\vec{r}_1) - e^2 \sum \limits_{1 < j} \alpha_n (|z_i - z_j| / L) + \text{cst} \]

and the Boltzmann factor is, at \( T = 2 \),

\[ \exp(-\beta H) = C \left[ \det \left( \exp[-V(\vec{r}_1)] \right) z_i^{j-1} \right]_{i,j=1,\ldots,N} \]

where \( C \) is some constant. The functions \( \exp[-V(\vec{r})] z_i^{j-1} \) are not necessarily orthogonal to one another. However, in the thermodynamical limit, they span the space \( E \); we can choose in this sapce an orthogonal basis \( \psi_j \), and rewrite the Boltzmann factor with a new determinant

\[ \exp(-\beta H) = C \left| \det \left( \psi_j(\vec{r}_1) \right) \right|^2. \]

It is then easy to see that the densities are related to the projector

\[ \langle \vec{r}_1 | P | \vec{r}_2 \rangle = \sum \limits_j \frac{\psi_j(\vec{r}_1) \psi_j(\vec{r}_2)}{\int d^2 \vec{r} \left| \psi_j(\vec{r}) \right|^2} \]

as stated above. The whole formalism much resembles the quantum-mechanical description of a system of independent fermions in an external field.
This new approach to the one-component plasma allows to make full use of the symmetries of the problem when choosing the basis $\psi_j$. A related important remark is that, for a given background charge density, the background potential $V(\mathbf{r})$ is not fully determined by Poisson’s equation, and this freedom can be used for choosing $V(\mathbf{r})$ in the most convenient way, depending on the problem to be solved.

For instance, for a constant background charge density $-e\rho_B$, it is convenient to work with circular symmetry, choosing $8\pi^2 V(\mathbf{r}) = \rho_B \mathbf{r}^2$; the functions $\exp[-V(r)] \mathbf{r}^{l-1}$ are orthogonal, and one recovers at once the known Gaussian correlation function $\rho^{(2)}(\mathbf{r}) = -\rho_B^2 \exp(-\rho_B r^2)$. It may however be more appropriate to choose $V(\mathbf{r}) = V(x)$, as illustrated by the following two examples.

3.1. Electrode with adsorption sites
The model under consideration mimics the electrical double layer at an electrode-electrolyte interface, the crystalline structure of the metallic electrode being taken into account. The present model has been previously studied by very difficult graph resummations. The new method is easier and gives more general results.

Let us assume that the interface is along the y axis. When there are no adsorption sites, the potential created by the background and the walls can be chosen of the form $e^2 V_0(x)$, and an orthogonal basis in the space $E$ can be defined by the function $\psi_k(\mathbf{r}) = \exp[-V_0(x) + k(x + iy)]$, with $k \in \mathbb{R}$; these functions are indeed orthogonal because of the factor $\exp(iky)$. Incidentally, one can recover in this way all the known results about inhomogeneous backgrounds with a charge density of the form $-e\rho_B(x)$.

Let us now add a line of equidistant adsorption sites along the y axis, creating an adsorption potential of the Baxter type, i.e.

$$\exp(-8V_{ad}) = 1 + \lambda \delta(x) \sum_m \delta(y - mb),$$

where $b$ is the distance between adjacent sites. It is convenient to write $k = 2\pi \frac{(\xi + n)b}{b}$, with $\xi \in [0,1]$ and $n$ integer. In presence of the adsorption potential, the functions $\exp(-8V_{ad}/2)\psi_k$ are not orthogonal. However, after these functions have been properly normalized, the matrix of their scalar products takes the simple form

$$M(\xi, n; \xi', n') = \delta(\xi - \xi')[\delta_{nn'} + \mu N(n) N(n')].$$

It is a simple matter to complete the diagonalization of this matrix which has a separable form, and to obtain explicit expressions for the projector $P$, the densities, and the correlations, both for an externally charged hard wall and
for an impermeable polarizable membrane.

Here too, the basic electrochemical quantities can be computed. A generalized
contact theorem has been verified: the kinetic pressure on the interface is
increased by a contribution from the gradient of the mobile charge density at
an adsorption site.

3.2. One-component plasma in a doubly periodic background

This model can be understood as made of mobile (classical) "electrons" inter-
acting between themselves and with a lattice of extended fixed "ions" (the back-
ground). It can also be regarded as a two-component plasma in which the
particles of one species have been frozen into a lattice.

The background density is the sum of its average value $\rho_0$ plus a periodic
modulation; if the lattice unit cell is a rectangle of sides $a$ (along $x$) and $b$
(along $y$), $\rho_0ab = 1$, i.e. there is one unit ionic charge per unit cell. The
background potential can be chosen of the form

$$e^2V(\vec{r}) = e^2\left[\rho_0 x^2 + \varphi(\vec{r})\right],$$

where $\varphi(\vec{r})$ has the double periodicity of the background. In the functional space
$E$, one can start with a basis defined by functions $\psi_k$ which are the same as
in section 3.1., except for a different normalization:

$$\psi_k(\vec{r}) = \exp\left[-i\varphi(\vec{r})\right] \exp\left[-\frac{\pi}{\rho_0 a} (x - \frac{k}{2\rho_0})^2 + iky\right].$$

Again, $k = 2\pi(x+n)/b$, with $x \in [0,1]$ and $n$ integer. These functions $\psi_k$ are not
orthogonal. However, since they depend on $x$ only through $x-na$, this suggests
introducing Bloch's functions

$$\tilde{\psi}_{\epsilon,n}(r) = \frac{1}{\rho_0} \exp\left(-2\pi i \epsilon n\right) \psi_{\epsilon,n}(\vec{r}),$$

and these functions $\tilde{\psi}$ do indeed form an orthogonal basis. From the functions $\tilde{\psi}$
one can build the projector $P$, the densities, and the correlations. The method
can be generalized to more complicated lattices.

The simplest example is for a square lattice ($a=b$) with a background
potential modulation such that

$$\exp\left[-2\varphi(\vec{r})\right] = 1 + \lambda(\cos 2\pi x + \cos 2\pi y),$$

where $X = x/a$, $Y = y/a$. Then, the one-particle density has the double integral
representation

$$\nu(\vec{r}) = \langle \vec{r} | P | \vec{r} \rangle \rho_0 \sqrt{2}\exp\left[-2\varphi(\vec{r})\right] \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta$$

$$\times \frac{\exp[-\pi(X-x)^2-\pi(Y-y)^2] \cos[2\pi(X-x)(Y-y)]}{1 + \lambda[\exp(-\pi/2)] (\cos 2\pi x + \cos 2\pi y)}.$$
Whatever the oscillations of the background may be, the correlations are found to decay exponentially and the Stillinger-Lovett sum rule is satisfied: The present model, considered as a two-component plasma with one species fixed on a lattice, is in its conducting phase at $\Gamma = 2$. This is in agreement with the arguments\textsuperscript{17} indicating that the Kosterlitz-Thouless transition occurs at $\Gamma \geq 4$ for this fixed-ion model also.

4. CONCLUSION

We have now reached a rather good understanding of both the one-component and the two-component plasmas, in two dimensions, at $\Gamma = 2$. After the difficulty about the divergence in the two-component plasma has been circumvented, the two-component plasma turns out to be the simplest one. A variety of inhomogeneous situations can be dealt with. The key of the solvability is the transformation of a problem about classical interacting particles into a problem about quantum-mechanical non-interacting particles, possibly in an external field.

REFERENCES


3) F. Cornu, Thèse n°887 (Orsay, 1989).


