Exact Results for the Two-Dimensional One-Component Plasma

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At some special temperature \( T_0 \), the distribution functions of a two-dimensional one-component plasma are explicitly computed up to the four-body one. The correlations have a Gaussian falloff. The distribution functions at \( T_0 \) are used for building a temperature expansion around \( T_0 \).

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A one-component plasma is a system of \( N \) identical particles of charge \( e \) embedded in a uniform neutralizing background of opposite charge. In two dimensions, the Coulomb interaction potential between two particles at a distance \( r \) from one another is

\[
e^{-q(r/L)} = -e^2 \ln(r/L),
\]

where \( L \) is a length scale. If one assumes the particles to be confined in a disk of radius \( R \), the total potential is

\[
V = -3N\pi e^2 \ln R / 2 N + \frac{N e^2}{2} \sum_i \left( \frac{\tilde{x}_i}{R} \right)^2
\]

\[= -e^2 \sum_{i,j} \ln \frac{\bar{r}_{ij}}{R},
\]

where \( \tilde{x}_i \) is the position of particle \( i \) (the origin is chosen at the center of the disk). By using the scaled variables \( \tilde{Z}_i = N^{1/2} \tilde{x}_i / R \), one easily shows that the excess free energy per particle, which must have a well-behaved thermodynamic limit, is necessarily of the form

\[
F_{\text{exc}} / N = -4e^2 \ln(\pi \rho L^2) + f(T),
\]

where \( \rho = N / \pi R^2 \) is the number density, and \( f(T) \) is some function of the temperature alone. Therefore, the equation of state has the simple form

\[
\rho = (k_B T - 4e^2 / \pi) \rho,
\]

where \( k_B \) is Boltzmann’s constant and \( T \) is the temperature. More information can be obtained at the special temperature \( T_0 = e^2 / 2k_B \); recently, the free energy has been exactly computed at \( T_0 \), with the result

\[
F_{\text{exc}} / N = -4e^2 \ln(\pi \rho L^2) + e^2 \left[ \frac{1}{2} - \frac{1}{3} \ln(2\pi) \right].
\]

In the present Letter, the distribution functions at \( T_0 \) are explicitly computed up to the four-body one and used for building a temperature expansion around \( T_0 \).

Using for \( \tilde{Z}_i \) polar coordinates \((z_i, \theta_i)\), one obtains at \( T_0 \) from (2) a Boltzmann factor

\[
\exp(-V/k_B T_0) = A \exp(-\sum_i z_i^2) \prod (Z_i - Z_j)^2,
\]

where \( A \) is a constant and \( Z_i = z_i \exp(\theta_i) \). This expression (6), which also occurs in the theory of random matrices, can be integrated upon variables \( \tilde{Z}_i (0 \leq z_i \leq N) \) by expanding the Vandermonde determinant \( \prod (Z_i - Z_j) \). One obtains the partition function

\[
\gamma(j, N) = A \pi^N N! \prod_{j=1}^{N} \gamma(j, N),
\]

where

\[
\gamma(j, N) = \int_0^\infty \exp(-z^2)2^{(j-1)2z} dz
\]

is the incomplete gamma function; in Ref. 4, (8) was used for computing the free energy (5). One can also obtain the \( n \)-body distribution functions

\[
\phi(1, \ldots, n)
\]

\[
\gamma(1, \ldots, n) = \exp(-\sum_{i=1}^{N} z_i^2) \text{Det} [K_n(z_i, Z_{i*})]_{i,j=1, \ldots, N},
\]

where

\[
K_n(Z_i, Z_{i*}) = \sum_{i=1}^{N} \frac{(Z_i Z_{i*})^{l-1}}{\gamma(l, N)}.
\]

In the thermodynamic limit \( N \to \infty \), \( \gamma(l, N) \to (l - 1)! \), and \( K_n(Z_i, Z_{i*}) \to \exp(Z_i Z_{i*}) \) [the terms with \( l \) close to \( N \) make no trouble since \( \gamma(N, N) \to 1/2(N - 1)! \)]. In this limit, one obtains from (10) the following explicit distribution functions: The one-body density \( \phi(1) = \rho \gamma(1) \) has the constant value \( \rho \). The pair distribution function is

\[
\phi(1, 2) = 1 - \exp(-\pi \rho \beta z^2),
\]

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where \( \overline{F}_1 = \overline{F}_2 - \overline{F}_1 \). The three-body distribution function is

\[
g(1,2,3) = 1 - \exp(-\pi \rho r_{12}^2) - \exp(-\pi \rho r_{23}^2) - \exp(-\pi \rho r_{31}^2) + 2 \exp(-\frac{1}{2} \pi \rho (r_{12}^2 + r_{23}^2 + r_{31}^2)) \cos[2 \pi \rho A(1,2,3)],
\]

where \( A(1,2,3) \) is the area of the triangle formed by particles \( (1,2,3) \). The four-body distribution function is

\[
g(1,2,3,4) = 1 - \exp(-\pi \rho r_{12}^2) - \cdots
+ \exp(-\pi \rho (r_{12}^2 + r_{34}^2)) + \cdots
+ 2 \exp(-\frac{1}{2} \pi \rho (r_{12}^2 + r_{23}^2 + r_{34}^2)) \cos[2 \pi \rho A(1,2,3)] + \cdots
- 2 \exp(-\frac{1}{2} \pi \rho (r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{41}^2)) \cos[2 \pi \rho A(1,2,3,4)] - \cdots,
\]

where

\[
A(1,2,3,4) = \frac{1}{2} |\overline{F}_1 \times \overline{F}_2|,
\]

is the area of the quadrilateral formed by particles \( (1,2,3,4) \) (or a difference of areas if that quadrilateral has intersecting sides).

These distribution functions are translationally invariant. They show neither long-range order nor quasi-long-range\(^7\) order: At \( T_0 \), the system is a fluid. A somewhat surprising result is that the correlations have a Gaussian falloff rather than the exponential one which is found\(^6\) in the high-temperature Debye approximation. One easily checks that the distribution functions obey the perfect screening and other sum rules.\(^10\)

It is convenient to express the temperature \( T \) through the dimensionless coupling constant \( \Gamma = e^2 / k_B T \); at \( T_0 \), \( \Gamma = 2 \). Using the distribution functions at \( \Gamma = 2 \), one can now build expansions in powers of \( \Gamma - 2 \). The pair distribution function at \( \Gamma \) is

\[
g(1,2;\Gamma) = g(1,2) + (\Gamma - 2) \{ -g(1,2) v(1,2) - 2 \rho \int [g(1,2,3) - g(1,2)] v(1,3) \, d3
- \frac{1}{2} \rho^2 \int [g(1,2,3,4) - g(1,2) g(3,4) - g(1,2,3) - g(1,2,4) + 2 g(1,2)]
\times v(3,4) \, d3 \, d4 \} + \cdots,
\]

where the absence of \( \Gamma \) in the arguments of \( g \) means \( \Gamma = 2 \). Equation (16) differs from the usual perturbation expansion for neutral fluids\(^11\) in two ways. First, there are additional terms due to the particle-background interaction. Second, as a consequence of perfect screening, there is no term of order \( 1/N \) in \( g(1,2,3,4) - g(1,2) g(3,4) \) when the pairs \( (1,2) \) and \( (3,4) \) are widely separated [for neutral fluids, such terms give additional finite contributions to (16) in the thermodynamic limit]. One finds from (16)

\[
g(\nu;\Gamma) = 1 - \exp(-\pi \rho r^2) + (\Gamma - 2) \{ -\exp(-\pi \rho r^2) [\ln(\pi \rho r^2) + C]
+ Ei(-\pi \rho r^2) - \frac{1}{2} Ei(-\frac{1}{2} \pi \rho r^2) + \frac{1}{2} \exp(-\pi \rho r^2) Ei\left(\frac{3}{2} \pi \rho r^2\right)\} + \cdots,
\]

where \( C = 0.5772\ldots \) is Euler's constant and \( Ei \) is the experimental-integral function. It can be checked that \( g(\nu;\Gamma) \) obeys the usual sum rules\(^10\) to order \( \Gamma - 2 \).

At small \( r \), for any value of \( \Gamma \), \( g(\nu;\Gamma) \) should exhibit a bare-potential factor \( \exp(\Gamma \ln r) \); this factor gives a monotonically behavior both in the Debye approximation\(^8\) valid near \( \Gamma = 0 \) and in the present expansion (17) around \( \Gamma = 2 \). At large \( r \), the bracket in (17) behaves like \( 2 \exp(-\frac{1}{2} \pi \rho r^2) / \pi r^2 \); thus, the correction of order \( \Gamma - 2 \) introduces a correction which is still Gaussian-like at long distance, but with a longer range. Further corrections of increasing order in \( \Gamma - 2 \) would have increasing ranges; presumably, if the whole series could be summed, one would recover near \( \Gamma = 0 \) the exponential falloff of the Debye approximation. Conversely, if the full Debye expansion around \( \Gamma = 0 \) could be summed, it should give a Gaussian at \( \Gamma = 2 \).

For large values of \( \Gamma \), one expects oscillations in \( g(\nu) \). Since the correction of order \( \Gamma - 2 \) in (17) has the sign of \( \Gamma - 2 \), it is tempting to conjecture that \( g(\nu) \) changes from a monotonic to an oscillating behavior precisely at \( \Gamma = 2 \).

Knowing \( g(\nu;\Gamma) \) up to the order \( \Gamma - 2 \), one can compute the free energy up to the order \( (\Gamma - 2)^2 \),
Quantum-Statistical Metastability

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Consider a system rendered unstable by both quantum tunneling and thermodynamic fluctuation. The tunneling rate $\Gamma$, at temperature $\beta^{-1}$, is related to the free energy $F$ by $\Gamma = (2/\hbar) \text{Im} F$. However, the classical escape rate is $\Gamma = (\beta \dot{\omega} / 2 \pi) \text{Im} F$, $-\dot{\omega}$ being the negative eigenvalue at the saddle point. A general theory of metastability is constructed in which these formulas are true for temperatures, respectively, below and above $\beta \hbar / 2 \pi$ with a narrow transition region of $O(\hbar^{1/2})$.

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Consider a system with a localized metastable ground state and a saddle point through which the system can escape to the true ground state. A simple example is a particle in the one-dimensional (1D) potential of Fig. 1. I shall at first concentrate on this example and then generalize to an arbitrary system (which may be a field theory). One may safely assume that both the ground-state energy, $\frac{1}{2} \hbar \omega_0 \left[ V''(x_0) = \omega_0^2 \right]$, and the temperature are small compared to the barrier height, $V_0$; otherwise, the system would not be metastable.

At temperatures small compared to $\hbar \omega_0$ the particle is mainly in the low-lying metastable states. These have wave functions that vanish at $-\infty$, are standing waves normalized to 1 in the well, and give an exponentially small probability current, $J_x$, at positive $x$, which is identified with the decay rate, $\Gamma(E)$. The nonconservation of $J_x$ requires that $E$ have an (exponentially small) imaginary part which obeys $\Gamma = (2/\hbar) \text{Im} E$. Taking a Boltzmann average of $\Gamma(E)$, we find $\Gamma = (2/\hbar) \text{Im} F$, to lowest order in exponentially small quantities.

At temperatures large compared to $\hbar \omega_0$ (but still small compared to $V_0$) we would expect classical thermodynamic fluctuations to dominate. The classical rate is calculated by setting up a