A kinetic theory approach to adiabaticity shortcuts and accelerated thermalization

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The executive summary

real space

phase space

Adiabatic opening
The executive summary

Adiabatic opening

Shortcut to adiabaticity protocol
GOAL: turn a slow process into an arbitrarily fast one
Transform an atomic cloud from an equilibrium state to a different equilibrium state

Framework: Boltzmann equation for a dilute elastic interacting gas
But question is somewhat broader!
The Boltzmann equation...

140 years ago

Transport in solids, energy transfer in plasmas, space shuttle aerodynamics (DSMC), neutron transport, granular gases etc

From a leap of faith to a branch of Mathematics
\[ \mathcal{H}(t) = \int f \log f \, dr \, dv \]

with \( \int f \, dr \, dv = 1 \)

- \( H(t) \) always decreases due to collisions
- The first Lyapunov function \( f(\mathbf{r}, \mathbf{v}, t) \)
- Yields time \( \rightarrow \), bridging micro dyn. to macro irrev.
- IMPORTANT: at long times, \( \log f \) is a collisional invariant...
- … should exhaust all conserved quantities

\[ \rightarrow f(\mathbf{r}, \mathbf{v}, t) = \exp \left\{ -\alpha - \beta v^2 - \gamma \cdot \mathbf{v} \right\} \]

1872: monoatomic; polyatomic more recent
Low density gas: $N$-body $\rightarrow$ 1 body dynamics

\[ f(r, v, t) \rightarrow \text{free streaming + binary collision} \]

\[
(\partial_t + v \cdot \nabla_r + F(r, t) \cdot \nabla_v) f = I_{\text{coll}}[v | f, f] 
\]

\[ F = -\nabla_r V \]

Equilibrium reads: \[ f \propto \exp(-2\beta V - \beta v^2) \]
Boltzmann himself realized that for special potentials \((V \propto r^2)\) equilibrium not reached

Not so well known, and considered as an uninteresting curiosity

"...for special outside potentials for instance the harmonic potential the spatial equilibrium distribution will not be reached in time. For such special potentials there are a host of special solutions of the Boltzmann equation (...) where the (coefficients) can be functions of space and time.(...) They have however only a limited interest"

What Uhlenbeck (and Boltzmann) did not know

- Non harmonic static traps are admissible surprisingly missed by all textbooks
- Can be generalized to time-dependent confinement
- Opens the way to 'reverse engineering' control. Connect two equil solutions in an arbitrary short time-span, $\ll$ time dictated by thermodynamic adiabaticity criterion
- $\rightarrow$ short-cuts to adiabaticity
What do these solutions look like?

Start with the direct view: static confinement

Swing-like mechanism through coupling term

\[ f(r, v, t) = \exp \left\{ -\alpha - \beta v^2 - \gamma \cdot v \right\} \]

Perpetual conversion kinetic/potential energy (or perpetual expansion)

\[ \rightarrow \alpha, \beta, \gamma \text{ are oscillating functions} \]

\[ \rightarrow \text{breathing modes (monopolar), undamped} \]

Not limited to small oscillations

How to proceed to find them?
The method

- Enforce validity of generalized Gaussian ansatz
  \[
  f(r, v, t) = \exp\{-\alpha - \beta v^2 - \gamma \cdot v\}
  \]

- Collisional integral = 0
  \[
  (\partial_t + v \cdot \nabla_r + F(r, t) \cdot \nabla_v) f = I_{\text{coll}}[v|f, f]
  \]
  → no dissipation

- Nullify free streaming
  \[
  \nabla_r \beta = 0, \\
  v^2 \partial_t \beta + v \cdot \nabla_r (\gamma \cdot v) = 0, \quad \text{for all } v, \\
  \partial_t \gamma + \nabla_r \alpha + 2\beta F = 0, \\
  \partial_t \alpha + F \cdot \gamma = 0.
  \]
The general solution

\[ \gamma(r, t) = \gamma_0(t) + J \wedge r - \dot{\beta} r \]

\[ \gamma_0(t) = 0 \quad \text{innocuous} \]

\[ J = 0 \quad \text{a bit less innocuous} \]
\[ \quad \text{but convenient} \]
\[ \quad \text{(conserved angular momentum)} \]

\[ \dot{\beta} (2 + r \cdot \nabla_r) V(r) + \dddot{\beta} r^2 / 2 = 0 \]

\[ F = -\nabla_r V \]

\[ V(r) = \frac{1}{2} \omega^2 r^2 + \frac{b}{r^2}, \]

\[ \dddot{\beta} + 4\omega^2 \dot{\beta} = 0, \quad \rightarrow \beta \text{ oscillates at } 2\omega. \]

So do \( \alpha \) and \( \gamma \).
Recent development: first 3D experimental evidence of breathers

Observation of a persistent non-equilibrium state in cold atoms

D. S. Lobser, A. E. S. Barentine, E. A. Cornell & H. J. Lewandowski

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From static to time-dependent traps

Consider
\[ V(r, t) = \omega^2(t) r^2 / 2 + b(t) / r^2 \]
or simply
\[ V(r, t) = \frac{\omega^2(t) r^2}{2} \]

Re-nullify free streaming from
\[ f(r, v, t) = \exp \left\{ -\alpha - \beta v^2 - \gamma \cdot v \right\} \]
\[ \rightarrow \quad \dddot{\beta} + 4 \omega^2 \dot{\beta} + 4 \omega \dot{\omega} \beta = 0 \]

\( \omega(t) \) versatile degree of freedom
can be controlled experimentally
(cold atoms, colloids, other settings ?)
What can we achieve?

Outline

1/ Parametric excitation - starter
2/ Reverse engineering view-point
2/ The 'reverse engineering' viewpoint

How should \( \omega(t) \) be designed to achieve a prescribed evolution....

… in particular one connecting 2 equilibrium states?

**Usually** : just be patient !

→ adiabatic evolution

\[
|\dot{\omega}| \ll \omega^2
\]

\[
|\dot{\omega}|/\omega \ll \tau_{\text{relax}}^{-1}
\]

\( E/\omega \) remains constant (\( E \) total mechanical energy)

Is that possible to short-cut this **long** waiting ?

→ **YES**
In other words: take two equilibrium states $(i$ and $f)$

They have to be connected by adiabatic transform

$$\frac{T_i}{\omega_i} = \frac{T_f}{\omega_f}$$

note $T \leftrightarrow 1/\beta$

Key equation

$$\dddot{\beta} + 4\omega^2 \dot{\beta} + 4\omega \dot{\omega} \beta = 0$$

For consistency: check slow evolution

$\rightarrow \beta(t)\omega(t)$ conserved
What we want to do: fast compression or expansion

Prescribed evolution $T(t)$ [or $\beta(t)$] in between

$$\dddot{\beta} + 4\omega^2\dot{\beta} + 4\omega\dot{\omega}\beta = 0$$

All dynamics allowed (provided fulfil b.c.)

Choice: $\beta(t)$ polynomial with

$$\beta(0) = \beta_i, \dot{\beta}(0) = 0, \ddot{\beta}(0) = 0, \dddot{\beta}(0) = 0, \beta(t_f) = \beta_f$$

$$\dot{\beta}(t_f) = 0, \ddot{\beta}(t_f) = 0, \text{ and } \dddot{\beta}(t_f) = 0$$

$$\beta(s) = (\beta_f - \beta_i)(-20s^7 + 70s^6 - 84s^5 + 35s^4) + \beta_i$$

with $s = t/t_f$
Hence fast expansion / compression without final excitation

5 fold decompression

quasi-adiabatic
$E/\omega$ constant

critical

expulsive trapping : $\omega^2 < 0$. 
Quantum formulation

\[ f(r, \nu, t) = \left( \epsilon + e^{-\alpha(r,t) - \beta(t)v^2 - \gamma \cdot \nu + \mu} \right)^{-1}, \]

where \( \epsilon = +1 \) for fermions and \( \epsilon = -1 \) for bosons.

The rest is unaffected...
Does it work?

Mathematically correct solutions...
Stability?

Perform DSMC simulations
*Direct Simulations of Monte Carlo* type
→ particle based method to solve Boltzmann eq.
(much use in aeronautics)

Columbia crash, 2003

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The roadmap

Choose a target $T(t)$ evolution

Solve

$$\dddot{\beta} + 4 \omega^2 \dot{\beta} + 4 \omega \dot{\omega} \beta = 0$$

possible analytically:

$$\omega^2(t) = \frac{1}{\beta^2} \left\{ \omega_i^2 \beta_i^2 + \frac{\beta^2}{4} - \frac{\beta \ddot{\beta}}{2} \right\}$$

Then perform Monte Carlo with such a trapping...

… and compare simulation with target $T(t)$
Comparison to Monte Carlo

Solve 'exactly' Boltz. eq for a **hard sphere system**

Departure from adiabaticity
Going still faster

Excellent agreement Monte Carlo ↔ theory
Scaling form?

\[ \tilde{r} = \frac{r}{\sqrt{\beta}} \]

\[ \tilde{v} = \sqrt{\beta} v + \frac{\gamma}{2\sqrt{\beta}} \]

Embody the full time dependence

\[ f(r, v, t) \rightarrow \tilde{f}(\tilde{r}, \tilde{v}) \]
Discussion

Phase space manipulation technique works for **interacting systems** arbitrary short time scales...

In practice: implementation of repeller confinement? limited energy budget (becomes costly)?

Room for extensions: introduce optimality criterion e.g. choose $T(t)$ so that it minimizes energy cost of set-up, before finding $\omega(t)$.

Question: **effect of noise** on solutions found?

And beyond the Boltzmann limit (increasing density)?

Experimental realizations?
Collaboration with

David Guéry-Odelin (Toulouse)

+ Alexei Chepelianski (Orsay)
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For trapped colloids / Open systems

Thank you