Can the renormalization group lead to exact results?

Analytic results in statistical mechanics
My topic will be almost 50 years old, so it’s more history than science

’70 : two major breakthroughs based on the renormalization group

QCD and critical phenomena

Let us compare the two fields. In both cases there are qualitative and quantitative aspects.
(i) Among the reasons which led to QCD, the experimental observations of the internal structure of the proton played a central role. Deep inelastic electron scattering at SLAC in the sixties revealed that the constituents of the proton interact very weakly at short distance, in spite of being confined in the proton. When it became understood that non-abelian gauge theories were the only field theories which possessed the property of ”asymptotic freedom” i.e. a vanishing interaction at short distance between the quarks, a non-abelian gauge theory of strong interactions was immediately proposed. Although an SU(3) gauge group was favored, it waited more quantitative results to confirm this qualitative picture.
Again the deep inelastic experiments gave the missing quantitative clue. In QCD the logarithmic approach to vanishing coupling at high energy

$$\frac{1}{g^2} \simeq \log q^2$$

manifested itself in the moments of the structure functions which exhibit logarithmic deviations from a naive model of non-interacting quarks (see for instance D. Gross’ lectures, Les Houches 1975).
A. I, Larkin and D. E. Khmel’nikskii


Uniaxial Ferroelectrics with strong dipolar interactions

It contains three major breakthroughs

1. Meanfield theory is quantitatively exact above dimension four

2. At d=4 there are calculable logarithmic deviations from mean field theory
For instance the specific heat instead of having a simple jump at $T_c$, like in mean field theory, should slowly diverge as

$$C = A_{\pm}(|\log|\frac{T - T_c}{T_c}||)^{1/3}$$

with

$$\frac{A_+}{A_-} = \frac{1}{4}$$

Note that this RG prediction is exact, no $\epsilon$-expansion, $1/n$-expansion, real-space RG, etc.. It follows simply from

$$\frac{1}{g(\lambda)} \approx \log(1/\lambda)$$

when $\lambda \to 0$
3. If dipolar interactions are not negligible with respect to exchange forces then the four-dimensional theory applies to $d=3$.

Dimension three is accessible to experiments, but after making a crystal with strong dipolar forces, and an easy axis of magnetization (uniaxial means anisotropic crystal)

G. Ahlers, A. Kornblit and H. Guggenheim  
Phys. Rev. Lett. 34, 1227 (1975)

$\text{LiTbF}_4$ $T_c = 2.885K$
They tested

\[ C = A_\pm (| \log \left| \frac{T - T_c}{T_c} \right| )^{1/3} \]

\[ \frac{A_+}{A_-} = \frac{1}{4} \]

and found: The power of the leading logarithmic term is found to be 0.34 ± 0.03, and the corresponding amplitude ratio is 0.24 ± 0.01.

A wonderful test of RG, free of the usual approximate schemes which are not easy to control ... provided the correspondance

\[ d = 4 \text{ (shortrange)} \iff d = 3 \text{ (dipolar)} \]

is true??
Let us examine the basis for this correspondance in LK:

(i) SR Exchange forces

\[ J_{ij} \vec{S}_i \dot{\vec{S}}_j \]

or in Fourier space

\[ J(q) \vec{S}(q) \dot{\vec{S}}(-q) \]

and with \( J(q) = r + q^2 + \cdots \) a propagator

\[ G(q) = \frac{1}{q^2 + r} \]
(ii) Uniaxial dipolar

\[ J_{ij} S_i^z S_j^z + \alpha S_i^z \frac{\partial}{\partial z} S_j^z \frac{\partial}{\partial z} \frac{1}{r_{ij}} \]

hence

\[ G(q) = \frac{1}{\bar{q}^2 + \alpha \frac{q_z^2}{q^2} + r} \]

Corrections to mean field involve integrals such as

\[ \mathcal{I} = \int d^d q \ G^2(q) \]

in a bounded domain (Brillouin zone) near \( T_c \) (i.e. \( r \to 0 \))
(i) For SR forces

\[ I = \int d^d q \frac{1}{(q^2 + r)^2} \]
remains finite for \( d > 4 \), diverges logarithmically at \( d = 4 \)

(ii) For dipolar forces \( I \) remains finite for \( d > 3 \), diverges logarithmically at \( d = 3 \)

\[ I = 2\pi \int q^2 dq \sin \theta d\theta \frac{1}{(q^2 + \alpha \cos^2 \theta + r)^2} \]
looks like 4D with a fourth component \( q_4 = \alpha^{1/2} \cos \theta \)

\[ I = \frac{1}{2} \int d^4 Q \frac{1}{(Q^2 + r)^2} \]
Is that sufficient to assert that the correspondance

\[ d = 4 \text{ (shortrange)} \iff d = 3 \text{ (dipolar)} \]

is true?

- NO the correspondance is only qualitative
- YES the correspondance is valid at one-loop level
  ... and almost valid at two-loop level
• S.R. $d = 4$

$$C = A_{\pm} |\log |t||^{1/3}(1 - \frac{25 \log |\log |t||}{81 |\log |t||})$$

• Dipolar $d = 3$

$$C = A_{\pm} |\log |t||^{1/3}(1 - \frac{1}{243}(108 \log \frac{4}{3} + 41) \frac{\log |\log |t||}{|\log |t||})$$

Corrections are of order $O\left(\frac{1}{\log |t||}\right)$ with $t = \frac{T - T_c}{T_c}$
\[
\frac{25}{81} \approx 0.308
\]
\[
\frac{1}{243} \left(108 \log \frac{4}{3} + 41\right) \approx 0.296
\]

The correspondance is \emph{accidentally} nearly true also at two loop-order.

In practice a \( \frac{\log(\log)}{\log} \) is too slowly varying to be measurable, but this still provides one of the best tests of RG since it is free of any approximation.